II. *De Seriebus infinitis Tractatus. Pars Prima. Auctore* Petro Remundo *de* Monmort. R. S. S.

Prop. 1. *Prob.*

INvenire ſummam terminorum quot libuerit Seriei hujus $a \times \overline{a+n} \times \overline{a+2n} \times \&c. \times \overline{a+p-1n}$ $+ \overline{a+n} \times \overline{a+2n} \times \overline{a+3n} \times \&c. \times \overline{a+pn}$ $+ \overline{a+2n} \times \overline{a+3n} \times \overline{a+4n} \times \&c. \times \overline{a+p+1n}$ $+ \overline{a+3n} \times \&c.$ Ubi eſt n differentia data, tam inter Factores continuos, a, $a+n$, $a+2n$, *&c* ejuſdem cujuſvis termini, quàm inter Factores homologos terminorum diverſorum in Serie continuatâ; atque deſignat p numerum factorum hujuſmodi in quovis termino.

Solutio Per x deſignetur primus Factorum in ultimo terminorum quorum ſumma requiritur, atque ſumma illa erit

$$\frac{x \times \overline{x+n} \times \&c. \times \overline{x+pn} - \overline{a-n} \times a \times \&c. \times \overline{a+p-1n}}{\overline{p+1}\,n}$$

Q. E. I.

Ex. 1. Proponatur Series numerorum naturalium $1+2+3+4+\&c.$ & invenienda ſit ſumma tot terminorum quot ſunt unitates in numero z, qui in hoc caſu eſt etiam ultimus terminorum quorum ſumma requiritur. In hoc itaque caſu ſunt $a=1$, $n=1$, $p=1$, & $x=z$. Unde fit $x \times \overline{x+n} \times \&c. \times \overline{x+pn} = z \times \overline{z+1}$, $\overline{a-n} \times a \times \&c. \times \overline{a+p-1n} = 0 \times 1$, atque $\overline{p+1}\,n = 2 \times 1$; adeoque ſumma quæſita eſt $\frac{z \times \overline{z+1}}{2}$.

Ex. 2. Invenienda ſit ſumma tot terminorum, quot ſunt unitates in numero z, Seriei $1+3+6+10+\&c.$ Numerorum Triangularium. Numeri 1, 3, 6, 10, *&c.* in hac

Serie ſic ſcribi poſſunt $\frac{1\times2}{2}$, $\frac{2\times3}{2}$, $\frac{3\times4}{2}$, $\frac{4\times5}{2}$, &c. Hoc pacto, ſepoſito diviſore dato 2, Series revocatur ad formam Propoſitionis, exiſtentibus $a=1$, $n=1$, & $p=2$. $x=z$. Unde ſumma Seriei duplicata eſt $\frac{x\times\overline{x+1}\times\overline{x+2}-0\times1\times2}{3}=\frac{x\times\overline{x+1}\times\overline{x+2}}{3}$; adeoque habitâ ratione diviſoris 2, Summa Seriei ipſius eſt $\frac{x\times\overline{x+1}\times\overline{x+2}}{2\times3}$, vel $\frac{z\times\overline{z+1}\times\overline{z+2}}{2\times3}$, in hoc caſu exiſtente x eodem ac z. Ad eundem modum inveniuntur ſummæ cæterorum numerorum figuratorum, quorum formulæ jam vulgò innoteſcunt.

Ex. 3. Sint $a=1$, $n=2$, $p=3$, ut ſit Series propoſita $1\times3\times5+3\times5\times7+5\times7\times9+$ &c. In hoc itaque caſu formula ſummæ fit

$$\frac{x\times\overline{x+2}\times\overline{x+4}\times\overline{x+6}-\overline{1-2}\times1\times3\times5}{4\times2}=$$

$$\frac{x\times\overline{x+2}\times\overline{x+4}\times\overline{x+6}+15}{8}.$$

Verbi gratiâ, ſi quæratur ſumma decem terminorum, fit $x=19$ (nempe terminus decimus in Serie Arithmeticè proportionalium, 1, 3, 5, 7, &c.) adeoque ſumma eſt $\frac{19\times21\times23\times25+15}{8}$ $=28680$. Propoſitio vero ſic demonſtratur.

Demonſtratio. Sit Series quantitatum $A, B, C, D, E,$ &c. quarum differentiæ conſtituant Seriem $a, b, c, d,$ &c. (nempe ut ſint $a=B-A$, $b=C-B$, $c=D-C$, &c.) Hinc ſtatim colligitur eſſe $a+b=C-A$, $a+b+c=D-A$, $a+b+c+d=E-A$: & in genere aggregatum quotlibet terminorum Seriei $a, b, c, d,$ &c. æquale eſt termino proximè ſubſequenti Seriei $A, B, C, D, E,$ &c. mulctato termino primo A. Pro $A, B, C,$ &c. ſume terminos

$\frac{\overline{a-n}\times a\times\text{\&c.}\times\overline{a+p-1n}}{\overline{p+1n}}$, $\frac{a\times\overline{a+n}\times\text{\&c.}\times\overline{a+pn}}{\overline{p+1n}}$, $\frac{\overline{a+n}\times\overline{a+2n}\times\text{\&c.}\times\overline{a+p+1n}}{p+1n}$, *&c.* hoc eſt, valores ſucceſſivos ipſius $\frac{x\times\overline{x+n}\times\text{\&c.}\times\overline{x+pn}}{\overline{p+1n}}$; & eorum differentiæ, pro $a, b, c, d,$ *&c* ſumendæ, erunt $a\times\overline{a+n}\times\text{\&c.}\times\overline{a+p-1n}$, $\overline{a+n}\times\overline{a+2n}\times\text{\&c.}\times\overline{a+pn}$, *&c.* qui ſunt ipſiſſimi termini Seriei propoſitæ. Sed comparando has Series, ſi terminus aliquis Seriei poſterioris ſit $x\times\overline{x+n}\times\text{\&c.}\times\overline{x+p-1n}$, conſtat terminum uno ulteriorem in Serie priori fore $\frac{x\times\overline{x+n}\times\text{\&c.}\times\overline{x+pn}}{\overline{p+1n}}$. Summa itaque Seriei poſterioris uſque terminum $x\times\overline{x+n}\times\text{\&c.}\times\overline{x+p-1n}$ incluſivè eſt $\frac{x\times\overline{x+n}\times\text{\&c.}\times\overline{x+pn}-\overline{a-n}\times a\times\text{\&c.}\times\overline{a+p-1n}}{\overline{p+1n}}$

Q. E. D.

Scholium 1. In hâc propoſitione continetur particula quædam Methodi incrementorum, de quâ ante biennium librum edidit D. *Brook Taylor Soc. Reg. Lond. Secr.* mihi amicitiâ conjunctiſſimus. Librum ipſum adeat qui de eâ methodo plura ſcire velit: ad inſtitutum noſtrum ſufficit obſervare quanta interſit affinitas inter Methodum hanc & Methodum Fluxionum ſeu differentialem Nam ut in Methodo differentiali, ad inveniendum differentiale ipſius x dignitatis x^m, unum latus x convertendum eſt in differentiam dx; & ortum ducendum eſt in dignitatis Indicem m, ut ſit $m\,dx\,x^{m-1}$ differentiale quæſitum; ſic in Methodo Incrementorum *Ad inveniendum Incrementum facti hujuſmodi* $x\times\overline{x+n}\times\overline{x+2n}$, (*ubi factores* x, $x+n$,

$x+2n$, *ſunt in progreſſione Arithmeticâ, cujus differentia communis eſt ipſius x Incrementum datum n,) Factorum minimus x convertendus eſt in Incrementum, & ortum ducendum eſt in numerum Factorum, ut ſit* $3n\times\overline{x+n}\times\overline{x+2n}$ *Incrementum quæſitum, numero Factorum in caſu expoſito exiſtente* 3. Sic etiam ipſius $x\times\overline{x+n}$ Incrementum fit $2n\times\overline{x+n}$.

2. Incrementa etiam Reciprocorum hujuſmodi Factorum inveniuntur per eandem regulam; hoc nempe obſervato, quòd cum ſit Diviſio contrarium Multiplicationis, vice ablationis minimi Factorum, ſit jam addendus alius factor adhuc uno Incremento major; item quòd Factorum numerus ſit ſcribendus cum ſigno negativo. Hoc pacto ipſius $\frac{1}{x}$ Incrementum fit $\frac{-1\times n}{x\times\overline{x+n}}$; ipſius $\frac{1}{x\times\overline{x+n}}$ Incrementum fit $\frac{-2\times n}{x\times\overline{x+n}\times\overline{x+2n}}$; & ſic de aliis hujuſmodi. Hoc facile probatur ſumendo differentias inter Integralium valores duos continuos.

3. Inſiſtendo veſtigiis Methodi directæ, hinc colliguntur præcepta Methodi inverſæ, quibus inveniuntur Integralia Incrementorum oblatorum. *Applicetur enim Incrementum oblatum ad lateris Incrementum datum; addatur Factor adhuc uno Incremento minor, & applicetur ortum ad numerum Factorum ſic auctorum.* Sic *e. g.* oblato Incremento $n\times x\times\overline{x+n}\times\overline{x+2n}$. fit primò $x\times\overline{x+n}\times\overline{x+2n}$; deinde $\overline{x-n}\times x\times\overline{x+n}\times\overline{x+2n}$, addito Factore $\overline{x-n}$; denique $\frac{\overline{x-n}\times x\times\overline{x+n}\times\overline{x+2n}}{4}$, quod eſt Integrale quæſitum. Hoc quidem ubi Factores ſunt Multiplicantes; Ubi vero Factores occupant locum diviſoris, mutatis mutandis, regula hæc eſt, *Applicetur Incrementum oblatum ad lateris incrementum datum; rejiciatur*

Factorum maximus, & applicetur ortum ad numerum Factorum relictorum cum ſigno negativo. Exempli gratiâ oblato Incremento $\frac{n}{x\times\overline{x+n}\times\overline{x+2n}}$, fit primò $\frac{1}{x\times\overline{x+n}\times\overline{x+2n}}$, deinde $\frac{1}{x\times\overline{x+n}}$, denique $\frac{1}{-2\times x\times\overline{x+n}}$, ſeu $\frac{-1}{2x\times\overline{x+n}}$, quod eſt Integrale quæſitum.

4. In caſu hoc noviſſimo Integrale inventum, cum ſigno contrario, æquale eſt ſummæ omnium Incrementorum in Serie in infinitum continuatâ; *v. g.* eſt $\frac{1}{2x\times\overline{x+n}}$ $= \frac{n}{x\times\overline{x+n}\times\overline{x+2n}} + \frac{n}{\overline{x+n}\times\overline{x+2n}\times\overline{x+3n}}$ $+ \frac{n}{\overline{x+2n}\times\overline{x+3n}\times\overline{x+4n}} +$ *&c.* Nam in hoc caſu, facto x tandem infinito, evaneſcit $\frac{1}{2x\times\overline{x+n}}$, hoc eſt, ultimus terminorum *A, B, C; &c.* fit nihil; & ob contrarietatem ſignorum Integralis & Incrementi, vice $-A$ exprimitur aggregatum per $+A$.

Lemma 1.

Per X deſignetur terminus quilibet in Serie quâvis numerorum *M, N, O, P, &c*; per x deſignetur locus termini iſtius X in Serie illâ (*v. g.* ut ſit $x=1$, quando deſignat X terminum primum M, ſit $x=2$, quando deſignat X terminum ſecundum N, & ſic de cæteris) & ſint terminorum *M, N, O, P* prima differentiarum primarum b, c prima differentiarum ſecundarum, d prima tertiarum, e prima quartarum, & ſic porrò. Tum erit

$X = M + b \times \frac{x-1}{1} + c \times \frac{x-1}{1} \times \frac{x-2}{2} + d \times \frac{x-1}{1} \times \frac{x-2}{2} \times \frac{x-3}{3} + e \times \frac{x-1}{1} \times \frac{x-2}{2} \times \frac{x-3}{3} \times \frac{x-4}{4} + \&c.$ Sequitur hoc ex tabulâ æquationum *pag.* 66. tractatûs noſtri *Eſſay d'Analyſe, &c.*

Lemma 2.

Iiſdem poſitis, per z deſignetur terminus quilibet in Serie Arithmeticè proportionalium a, $a + n$, $a + 2n$, *&c.* & ſit jam $X = A + Bz + Cz \times \overline{z+n} + Dz \times \overline{z+n} \times \overline{z+2n} + Ez \times \overline{z+n} \times \overline{z+2n} \times \overline{z+3n} + \&c.$ Tum ipſorum A, B, C, D, E, *&c.* valores erunt.

$$A = M + b \times \frac{-a}{n} + c \times \frac{-a}{n} \times \frac{-a-n}{2n} +$$

$$+ d \times \frac{-a}{n} \times \frac{-a-n}{2n} \times \frac{-a-2n}{3n} +$$

$$+ e \times \frac{-a}{n} \times \frac{-a-n}{2n} \times \frac{-a-2n}{3n} \times \frac{-a-3n}{4n} + \&c.$$

$$B = \frac{1}{n} \times b + c \times \frac{-a-n}{n} + d \times \frac{-a-n}{n} \times \frac{-a-2n}{2n}$$

$$+ e \times \frac{-a-n}{n} \times \frac{-a-2n}{2n} \times \frac{-a-3n}{3n} \ \&c.$$

$$C = \frac{1}{n} \times \frac{1}{2n} \times c + d \times \frac{-a-2n}{n} + e \times \frac{-a-2n}{n} \times \frac{-a-3n}{2n} + \&c.$$

$$D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} d + e \times \frac{-a-3n}{n} + \&c.$$

$$E = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \times \frac{1}{4n} e + \&c.$$

Ordo formandi coefficientes ipſorum b, c, d, e, &c. in his valoribus, per ſe eſt ſatis manifeſtus.

Demonſtratio. Quoniam per x & z deſignantur termini correſpondentes progreſſionum Arithmeticarum 1, 2, 3, 4, &c & a, $a+n$, $a+2n$, $a+3n$, &c. indicabit $x-1$ numerum differentiarum n qui in z continetur, ut ſit $z=a+\overline{x-1}\,n$. Hinc ſit $x-1=\frac{z-a}{n}$, $x-2=\frac{z-n-a}{n}$, $x-3=\frac{z-2n-a}{n}$, &c. Subſtituendo itaque hos valores $x-1$, $x-2$, $x-3$, &c. in Serie Lemmatis præcedentis, & terminis in ordinem redactis, prodeunt ipſorum A, B, C, &c. valores exhibiti.

Cor. Ubi $a=n$, prodeunt A, B, C, D, &c. per formulas ſimpliciores, nempe

$$A=M-b+c-d+e\ \&c.$$

$$B=\frac{1}{n}\times\overline{b-2c+3d-4e\ \&c.}$$

$$C=\frac{1}{n}\times\frac{1}{2n}\times\overline{c-3d+6e\ \&c.}$$

$$D=\frac{1}{n}\times\frac{1}{2n}\times\frac{1}{3n}\times\overline{d+4e\ \&c.}$$

Lemma 3.

Symbolis X & x eodem modo interpretatis ac in Lemmate primo, ſint q, r, s, t, u, &c. generatores Trianguli Arithmetici cujus lineam tranſverſam, occupat Series M, N, O, P, Q, &c in ordine nempe inverſo, ut ſit q $(=M)$ generator ultimus, r penultimus, s antepenultimus, & ſic porrò. Tum erit

$$X=q+r\times\frac{x-1}{1}+s\times\frac{x-1}{1}\times\frac{x}{2}+t\times\frac{x-1}{1}\times\frac{x}{2}\times\frac{x+1}{3}$$
$$+\ \&c.$$

Conſtat ex contemplatione ipſius Trianguli Arithmetici, quam exhibuimus *pag.* 63 tractatûs *Eſſay d'Analyſe, &c.* ubi idem fuſius explicatur.

Lemma 4.

Iiſdem poſitis, & Symbolo z eodem modo interpretato ac in *Lem.* 2. ſi ſit $X = A + Bz + Cz \times \overline{z+n} + \&c.$ ut in *Lem.* 2. erunt coefficientium $A, B, C, D, \&c.$ valores.

$$A = q + r \times \frac{-a}{n} + s \times \frac{-a}{n} \times \frac{-a+n}{2n}$$

$$+ t \times \frac{-a}{n} \times \frac{-a+n}{2n} \times \frac{-a+2n}{3n} + \&c.$$

$$B = \frac{1}{n} \times r + s \times \frac{-a}{n} + t \times \frac{-a}{a} \times \frac{-a+n}{2n} + \&c.$$

$$C = \frac{1}{n} \times \frac{1}{2n} \times s + t \times \frac{-a}{n} + \&c.$$

$$D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \times t + \&c.$$

Ordo coefficientium in his valoribus eſt manifeſtus, & demonſtratur Lemma ad modum *Lemmatis* 2.

Cor. 1. Ubi $a = n$, coefficientes, $A, B, C, D, \&c.$ prodeunt per formulas ſimpliciores, nempe

$$A = q - r, \qquad C = \frac{1}{n} \times \frac{1}{2n} \times \overline{s-t}$$

$$\&c.$$

$$B = \frac{1}{n} \times \overline{r-s}, \qquad D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n}\,\overline{t-u}$$

Cor. 2. Unde ſi generatorum $q, r, s, t, u, \&c.$ aliquot ſint inter ſe æquales, exhibebitur X per formulam ſimpliciorem, evaneſcentibus aliquot coefficientium $A, B, C, D, \&c.$

Sic exempli gratiâ, propoſitâ Serie numerorum 4, 69, 530, 2676, 10350, *&c.* qui conſtituunt lineam decimam tranſverſam in Triangulo Arithmetico cujus generatores tres priores ſunt 54, — 18, 5, & ſeptem poſteriores ſunt æquales 4; exiſtente $a = 1 = n$, Terminus X exhibetur per formulam quatuor tantùm terminorum.

$$-\frac{z}{1}.\frac{z+1}{2}.\frac{z+2}{3}\,\&c.\times\frac{z+6}{7}+23\,\frac{z}{1}.\frac{z+1}{2}.\,\&c.\times\frac{z+6}{7}-72\,\frac{z}{1}.\frac{z+1}{2}.\,\&c.\times\frac{z+7}{8}+54\,\frac{z}{1}.\frac{z+1}{2}.\,\&c.\times\frac{z+8}{9}.$$

evaneſcentibus coefficientibus ſex primis A, B, C, D, E, F.

Prop. II. *Prob.*

Invenire ſummam quotlibet terminorum Seriei

$$\frac{M}{a\times\overline{a+n}\times\&c.\times\overline{a+p-1\,n}}+\frac{N}{\overline{a+n}\times\&c.\times\overline{a+pn}}+\frac{O}{\overline{a+2n}\times\&c.\times\overline{a+p+1\,n}}+\&c.$$

ubi numeratores M, N, O, *&c.* conſtituunt Seriem quamlibet terminorum, quorum differentiæ, vel primæ, vel ſecundæ, vel aliæ quædam dantur; vel quod perinde eſt, qui conſtituunt lineam quamvis tranſverſam in dato quovis triangulo Arithmetico; Denominatores autem conſtituunt Seriem in *Prop.* I. exhibitam.

Solutio. Per X deſignetur primus factorum a, $a+n$, $a+2n$, &c. in denominatore ejuſdem termini, ut ſint X & z iidem ac in *Lemm:* præmiſſis, adeoque deſignetur terminus quilibet Seriei per $\frac{X}{z\times\overline{z+n}\times\&c.\times\overline{z+p-n}}$ Per *Lem.* 2, vel per *Lem.* 4. (prout magis commodum videatur vel differentias, vel generatores trianguli Arithmetici adhibere,) reſolvatur X in Multinomium $A + B \times z + Cz \times \overline{z+n} + Dz \times \overline{z+n} \times \overline{z+2n} + \&c.$ Hoc pacto (terminis multinomii ad denominatorem $z \times \overline{z+n} \times \&c. \times \overline{z+p-n}$, applicatis) terminus quilibet Seriei revocabitur ad formulam $\frac{A}{z \times \overline{z+n} \times \&c. \times \overline{z+p-1n}}$

$$+ \frac{B}{\overline{z+n} \times \&c. \times \overline{z+p-1n}} + \frac{C}{\overline{z+2n} \times \&c \times \overline{z+p-1n}}$$

$+ \&c.$

Unde (per Scholium 4 *Prop.* I.) aggregatum totius Seriei, à termino $\frac{X}{z \times \overline{z+n} \times \&c. \times \overline{z+p-1n}}$ incluſive in infinitum continuatæ, eſt

$$\frac{A}{\overline{p-1} \times n \times z \times \overline{z+n} \times \&c. \times \overline{z+p-2n}}$$

$$+ \frac{B}{\overline{p-2} \times n \times \overline{z+n} \times \&c. \times \overline{z+p-2n}}$$

$$+ \frac{C}{\overline{p-3} \times n \times \overline{z+2n} \times \&c. \times \overline{z+p-2n}} + \&c.$$

re ſi dematur hoc aggregatum ab ejuſdem aggregati valore quando $z = a$, reſiduum erit ſumma omnium terminorum ante terminum $\frac{X}{z + \&c.}$, hoc eſt, tot terminorum quot ſunt unitates in $\frac{z-a}{n}$. *Q. E. I.*

Ex. 1. Sit primum exemplum in Serie $\frac{5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}$

+

$$+\frac{41}{5\cdot7\cdot9\cdot11\cdot13\cdot15}+\frac{131}{7\cdot9\cdot11\cdot13\cdot15\cdot17}$$

$$+\frac{275}{9\cdot11\cdot13\cdot15\cdot17\cdot19}+\frac{473}{11\cdot13\cdot15\cdot17\cdot19\cdot21}$$

+ &c. Sunt hic $a=3$, $n=2$, $t=6$, $M=5$, & capiendo differentias numeratorum inveniuntur $b=36$, $c=54$, $d=0=e=$ &c. Hinc in Lemmate ſecundo ſunt $A=5+36\times\frac{-3}{2}+54\times\frac{-3}{2}\times\frac{-5}{4}=\frac{209}{4}$, $B=\frac{1}{2}\times36+54\times\frac{-5}{2}=\frac{-99}{2}$, $C=\frac{1}{2}\times\frac{1}{4}\times54=\frac{27}{4}$, $D=0=E=$ &c. Summa itaque totius Seriei eſt

$$\frac{209}{4\times5\times2\times3\cdot5\cdot7\cdot9\cdot11}+\frac{-99}{2\times4\times2\times5\cdot7\cdot9\cdot11}+\frac{27}{4\times3\times2\times7\cdot9\cdot11}=\frac{283}{80\times3\cdot5\cdot7\cdot9\cdot11},$$

atque ſumma terminorum numero $\frac{z-3}{2}$ $(=\frac{z-a}{n})$ eſt

$$\frac{283}{80\times3\cdot5\cdot7\cdot9\cdot11}-\frac{209}{40\times z.\overline{z+2}.\overline{z+4}.\overline{z+6}.\overline{z+8}}$$

$$+\frac{99}{16\times\overline{z+2}\times\overline{z+4}.\overline{z+6}.\overline{z+8}}-\frac{27}{24\times\overline{z+4}.\overline{z+6}.\overline{z+8}}$$

Quærantur *v. g.* octo termini; tum exiſtente $\frac{z-3}{2}=8$ fit $z=19$, quo valore in formulâ adhibito, prodit ſumma $\frac{155891}{2\cdot3\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot7\cdot11\cdot19\cdot23}$.

Iidem Numeratores occupant lineam tertiam tranſverſam in Triangulo Arithmetico

54 · 54 · 54 · 54 · 54 · 54 · &c.
− 18 . 36 . 90 . 144 . 198 . &c.
5 . 41 . 131 . 275 . &c.

Unde in formula *Lem.* 4. ſunt generatores $q = 5$, $r = -18$; $s = 54$, $t = 0 = \&c.$ & prodeunt coefficientes $A = 5 - 18 \times \frac{-3}{2} + 54 \times \frac{-3}{2} \times \frac{-3+2}{4} = \frac{209}{4}$, $B = \frac{1}{2} \times \overline{-18 + 54 \times} \frac{-3}{2} = \frac{-99}{2}$, $C = \frac{1}{2} \times \frac{1}{4} \times 54 = \frac{27}{4}$, $D = 0 = E = \&c.$ iidem ac ſupra.

Ex. 2. Sit Series $\frac{4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{69}{2 \cdot 3 \cdot \&c.\, 12} + \frac{530}{3 \cdot 4 \cdot \&c.\, 13} + \frac{2676}{4 \cdot 5 \cdot \&c.\, 14} + \frac{10350}{5 \cdot 6 \cdot \&c.\, 15} + \&c.$ Ubi ſunt $a = 1$, $n = 1$, $p = 11$, atque Numeratores conſtituant Seriem in *Corol.* 20. *Lem.* 4. exhibitam. Applicando itaque valorem X in *Corol.* illo ad denominatorem $z \times \overline{z+1} \times \&c. \times \overline{z+10}$, fit Seriei propoſitæ Terminus

$$\frac{-1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \times \overline{z+6} \cdot \overline{z+7} \cdot \overline{z+8} \cdot \overline{z+9} \cdot \overline{z+10}}$$
$$+ \frac{23}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \times \overline{z+7} \cdot \overline{z+8} \cdot \overline{z+9} \cdot \overline{z+10}}$$
$$- \frac{72}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \times \overline{z+8} \cdot \overline{z+9} \cdot \overline{z+10}}$$
$$+ \frac{54}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \times \overline{z+9} \times \overline{z+10}}.$$

Adeoque per hanc *Prop.* ſumma Seriei à termino illo in infinitum continuatæ eſt

$$\frac{-1}{4 \times 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \times \overline{z+6} \cdot \overline{z+7} \cdot \overline{z+8} \cdot \overline{z+9}}$$
$$+$$

$$+\frac{23}{3\times 1.2.3.4.5.6.7\times\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$-\frac{72}{2\times 1.2.3.4.5.6.7.8\times\overline{z+8}.\overline{z+9}}$$

$$+\frac{54}{1\times 1.2.3.4.5.6.7.8.9\times\overline{z+9}}.$$

Itaque pro z ſumpto 1, fit ſumma totius Seriei $\frac{305}{12\times 1.2.3.4.5.6.7.8.9.10}$. Et in genere ſumma terminorum numero $\frac{z-1}{1}$, eſt $\frac{305}{12\times 1.2.3.4.5.6.7.8.9.10}$

$$+\frac{1}{4\times 1.2.3.4.5.6\times\overline{z+6}.\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$-\frac{23}{3\times 1.2.3.4.5.6.7\times\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$+\frac{72}{2\times 1.2.3.4.5.6.7.8\times\overline{z+8}\times\overline{z+9}}$$

$$-\frac{54}{1\times 1.2.3.4.5.6.7.8.9\times\overline{z+9}}.$$

Scholium 1. In computandis ſummis hujuſmodi Serierum, calculus plerumque levior eſt adhibitis generatoribus trianguli Arithmetici, quam ſi adhibeantur differentiæ. Libet itaque hac occaſione oſtendere quomodo ex datis differentiis inveniri poſſunt generatores Trianguli Arithmetici.

Sunto itaque ω primus Seriei terminus, a differentia ultima data, b prima differentiarum penultimarum, c prima antepenultimarum, & ſic porrò d, e, *&c.* atque ſint t, u, x, y, *&c* generatores quæſiti Trianguli Arithmetici, cujus lineam tranſverſam ordine p occupet Series propoſita. Tum (quod ex contemplatione Trianguli Arithmetici facile conſtat) ſunt

$$a = t$$

$$b = \frac{p-1}{1}\,t + u$$

$$c = \frac{p-1}{1} \times \frac{p-2}{2}\,t + \frac{p-2}{1}\,u + x$$

$$d = \frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3}\,t + \frac{p-2}{1} \times \frac{p-3}{2}\,u + \frac{p-3}{1}\,x + y \text{ &c.}$$

Unde colliguntur generatorum valores

$$t = a$$

$$u = b - \frac{p-1}{1}\,t$$

$$x = c - \frac{p-1}{1} \times \frac{p-2}{2}\,t - \frac{p-2}{1}\,u$$

$$y = d - \frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3}\,t - \frac{p-2}{1} \times \frac{p-3}{2}\,u - \frac{p-3}{1}\,x \text{ &c.}$$

Ultimus autem generator æqualis eſt Seriei termino primo ω.

2. D$^{\text{nus}}$ de *Monſoury Abbas Orbacenſis* mihi amiciſſimus, & ruri vicinus, poſtquam cum eo hæc communicaveram, aliam invenit hujus Problematis Solutionem, cujus formulam ob ejus miram ſimplicitatem hic referre juvat. Itaque in Serie numeratorum ſint ω terminus primus, b prima differentiarum primarum, c prima ſecundarum, d prima tertiarum, & ſic porrò; atque ſit termini primi Denominator $z \times \overline{z+n} \times$ &c. $\times\, \overline{z+p-1\,n}$; Tum

ſumma totius Seriei in infinitum continuatæ exhibebitur per formulam

$$\frac{\omega}{n\times\overline{p-1}\times z\times\overline{z+n}\times\&c.\times\overline{z+\overline{p-2}\,n}}+\frac{b}{n^2\times\overline{p-1}\times\overline{p-2}\times\overline{z+n}\times\&c.\times\overline{z+\overline{p-2}\,n}}+$$

$$\frac{c}{n^3\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{z+2n}\times\&c.\times\overline{z+\overline{p-2}\,n}}$$

$+$ &c.

Sit exemplum in Serie $\frac{5}{3\cdot5\cdot\&c.\,13}+\frac{41}{5\cdot7\cdot\&c.\,15}+\frac{131}{7\cdot9\cdot\&c.\,17}+\frac{275}{9\cdot11\cdot\&c.\,19}+\&c.$ cujus ſummam jam exhibuimus. In hoc caſu ſunt $\omega=5$, $b=36$, $c=54$, $d=0=e=\&c.$ Unde per formulam ſumma Seriei integræ fit $\frac{5}{2\cdot5\times3\cdot5\ldots11}+\frac{36}{4\cdot5\cdot4\times5\ldots11}+\frac{54}{8\cdot5\cdot4\cdot3\times7\ldots11}=\frac{283}{80\times3\cdot5\ldots11}$, ut per formulam noſtram exhibetur. Si quæratur ſumma ejuſdem Seriei incipientis à termino decimo $\frac{2273}{21\ldots31}$, in eo caſu $\omega=2273$, $b=522$, $c=54$, & ſumma eſſet $\frac{2273}{2\cdot5\times21\ldots29}+\frac{522}{4\cdot5\cdot4\times23\ldots29}+\frac{54}{8\cdot5\cdot4\cdot3\times25\ldots29}$

Hæc formula eſt commodiſſima, & ſummam exhibet nullo ferè negotio, quoties quæritur ſumma Seriei integræ, & differentiæ non ſunt nimis multæ. Sed ubi plures ſunt differentiæ, & quæritur non Series integra, ſed termini tantùm initiales aliquammulti, formulæ noſtræ ſunt commodiores.

3. Quando Serierum termini formantur tantùm per Multiplicationem, nec afficiuntur diviſoribus variabilibus, ſummæ ſemper exhiberi poſſunt per Methodum in *Prop.* I. traditam, ſint licet formulæ quantumlibet compoſitæ. Nam poſſunt ſemper revocari ad terminos in formâ quam poſtulat Propoſitio illa. Sic ſi differentiæ ipſorum z & x ſint m & n, & deſignetur terminus Seriei per $z\,x$; hic terminus revocabitur ad formam $\overline{a - nz} + \frac{n}{m} z \times \overline{z + m}$; cujus Integrale datur *per Prop.* I; nempe quoniam $dx = n$, & $dz = m$, eſt $dx = dz \times \frac{n}{m}$; unde regrediendo ad integralia fit $x = \frac{n}{m} z + a$ (adjecto invariabili a, ut habeatur ratio relationis inter z & x in Seriei termino primo,) quod ſic ſcribi poteſt $\overline{a - n} + \frac{n}{m} \times z \overline{z + m}$, ut deinde in z ductum induat formam requiſitam. Et ad eundem modum procedere licet in aliis caſibus ejuſmodi. Sed ubi formulæ oblatæ diviſoribus afficiuntur, eædem ac in Calculo integrali, ut vocant, difficultates occurrunt, eâdem induſtriâ ſuperandæ. Nec tamen ſemper ſuperari poſſunt. Nam prætérquam quod vix certò ſciri poſſit quæ debeat relatio intercedere inter Numeratorem fractionis & Denominatorem, ut formula oblata ad Integrale revocari poſſit; ſæpe etiam difficillimum eſt explorare an adſit jam talis relatio in formulâ iſtâ, aut ſi deſit, an introduci poſſit. Quicquid ego in hâc materiâ potiſſimum inveni, continetur in tribus ſequentibus propoſitionibus.

Prop. III. *Prob.*

Creſcentibus, z, u, y, x, *&c.* per differentias datas n, m, l, o, *&c.* invenire valorem numeratoris integri N, ut exiſtente Denominatore $z.\overline{z+n}.\&c.\overline{z+pn} \times u.\overline{u+m}.\&c.\overline{u+qm} \times y.\overline{y+l}.\&c.\overline{y+rl} \times x.\overline{x+o}\ \&c.\ \overline{x+so}.\&c.$ Fractio ad Integrale revocari poſſit.

Solutio. Fiat $N = \overline{z+pn} \times \overline{u+qm} \times \overline{y+rl} \times \overline{x+so} \times \&c. - zuyx\ \&c.$ atque Integrale erit fractio, cujus Denominator $z.\overline{z+n}.\&c.\ \overline{z+\overline{p-1}n} \times u.\overline{u+m}.\&c.\overline{u+\overline{q-1}m} \times y.\overline{y+l}.\&c.\overline{y-\overline{x-1}l} \times x.\overline{x+o}.\&c\ \overline{x+\overline{s-1}o} \times \&c.$ exiſtente 1 Numeratore.

Differentia enim hujus fractionis eſt fractio cujus numerator eſt ipſius N valor exhibitus, & denominator idem eſt ac denominator propoſitus, ut fieri debuit.

Ex. 1. Sit denominator propoſitus $z \times \overline{z+2} \times u \times \overline{u+3}$. In hoc caſu ſunt $n=2$, $m=3$, $p=1$, $q=1$; adeoque eſt $N = \overline{z+2} \times \overline{u+3} - zu = 3z+2u+6$, & per $\dfrac{3z+2u+6}{z.\overline{z+2} \times u.\overline{u+3}}$ repreſentatur terminus Seriei ſummabilis, cujus nempe in infinitum continuatæ ſumma exhibetur per $\dfrac{1}{zu}$. Sint verbi gratiâ, ipſorum z & u primus valor communis 1, atque Series ſummabilis erit $\dfrac{11}{1.3 \times 1.4} + \dfrac{23}{3.5 \times 4.7} + \dfrac{35}{5.7 \times 7.10} + \&c$, quippe cujus totius ſumma eſt 1. Per p deſignetur ordo termini cujuſvis in hâc Serie, erit $p = \dfrac{z-1+2}{2} = \dfrac{u-1+3}{3}$, adeoque $z = 2p-1$, & $u = 3p-2$; quibus valoribus pro z & u ſcriptis, deſignabitur terminus per formulam $\dfrac{12p-1}{\overline{2p-1} \times \overline{2p+1} \times \overline{3p-2} \times \overline{3p+1}}$. Summa autem terminorum omnium ante terminum illum, hoc eſt terminorum initialium numero $\dfrac{z-1}{2} = p-1$, eſt

$1-\frac{1}{zu}, = \frac{zu-1}{zu}$, hoc eſt $\frac{6pp-7p+1}{\overline{2p-1}\times\overline{3p-2}}$. Quare pro p ſcripto $p+1$, erit $\frac{p\times\overline{6p+5}}{\overline{2p+1}\times\overline{3p+1}}$ aggregatum tot terminorum initialium quot ſunt unitates in p.

Ex. 2. Iiſdem manentibus z, u, n, m, ſit denominator $z.\overline{z+2}.\overline{z+4}\times u.\overline{u+3}$. Tum per formulam numerator erit $\overline{z+4}\times\overline{u+3}-zu=3z+4u+12$, & ſumma Seriei exhibebitur per formulam $\frac{1}{z.\overline{z+2}\times u}$. Sit ipſorum z & u primus valor communis 1, & hinc elicietur Series $\frac{19}{1.3.5\times 1.4}+\frac{37}{3.5.7\times 4.7}+\frac{55}{5.7.9\times 7.10}+$ *&c* $=\frac{1}{3}$.

Scholium. In Seriebus jam expoſitis eadem ubique eſt differentia inter factores continuos ejuſdem cujuſvis termini, ac inter factores homologos terminorum continuorum. In ſequentibus exempla quædam ſunt Serierum, quarum ſummæ in terminis numero finitis exhiberi poſſunt, quamvis ea regula non obſervetur.

Prop. IV. *Prob.*

Creſcente z per differentias datas qn, invenire numeratorem integrum N, ut ad Integrale revocari poſſit fractio, cujus Denominator ſit ex certo numero p terminorum z, $z+n$, $z+2n$, *&c.* Arithmeticè proportionalium in invicem ductorum. Debet autem eſſe q numerus integer minor quam factorum numerus p.

Solutio. Erit $N=\overline{z+\overline{p-1}\,n}\times\overline{z+\overline{p-2}\,n}\times$ *&c.* $\times\overline{z+\overline{p-q}\,n}-z\times\overline{z+n}\times$ *&c.* $\times\overline{z+\overline{q-1}\,n}$, Integrale exiſtente $\frac{1}{z \times \overline{z+n} \times \&c. \times \overline{z+\overline{p-q-1}\,n}}$ Demonſtratur ad modum propoſitionis præcedentis.

Sumptis ad libitum n, p, q, & primo valore z, hinc oriuntur infinitæ Series ſummabiles, cujuſmodi ſunt Series tres ſequentes.

$$A = \frac{5}{1.2.3.4} + \frac{9}{3.4.5.6} + \frac{13}{5.6.7.8} + \frac{17}{7.8.9.10} \ \&c.$$

$$B = \frac{1}{1.2.3.4.5} + \frac{4}{4.5.6.7.8} + \frac{9}{7.8.9.10.11} + \frac{16}{10.11.12.13.14} + \&c.$$

$$C = \frac{1}{1.2.3.4.5} + \frac{14}{5.6.7.8.9} + \frac{55}{9.10.11.12.13} + \frac{140}{13.14.15.16.17} + \&c.$$

Has Series jampridem communicavi cum primariis quibuſdam Geometris, à quibus minimè contemni videntur. Sic ad me ſcribit peritiſſimus Geometra D *Nicolaus Bernoulli* in epiſtolâ datâ 25 *Julii* 1716. " Vous " me ferez un extreme plaiſir, *Monſieur*, de me com" muniquer la Solution de voſtre probleme, *Etant donnée* " *une ſuitte des Fractions dont les Numerateurs ſoient des* " *nombres figurés quelconque*, & *dont les Denominateurs* " *ſoient formés du produit d'un nombre egal de Facteurs* " *qui ſoient en Progreſſion Arithmetique, trouver la ſom*" *me*; & principalement comment vous avez trouvè " ces deux formules $\frac{p}{24 \times \overline{4p+1}}$, $\frac{p.\overline{p+1}}{12 \times \overline{3p+1} \times \overline{3p+2}}$.

Hæ formulæ ſpectant ad Series C & B, deſignante p numerum terminorum, quorum ſumma requiritur. Sic etiam ad me ſcribit D. *Taylor* in epiſtola datâ 22 *Aug.* 1716. " Ut & quâ ratione incidiſti in ſummationem " Serierum à te exhibitarum, præſertim loquor de

" Serie $\frac{1}{1.2.3.4.5}+\frac{4}{4.5.6.7.8}+\frac{9}{7.8.9.10.11}+$ *&c.*
" quæ videtur eſſe altioris indaginis.

Sed ut ad exempla jam redeamus. In Serie *A* ſunt $p=4$, $q=2$, $n=1$, primo valore z exiſtente 1. Eſt itaque $\overline{z+3}\times\overline{z+2}-z\times\overline{z+1}=2\times\overline{2z+3}$ formula, unde (rejecto dato numero 2) derivantur numeratores 5, 9, 13, 17, *&c.* Formula etiam ſummæ eſt $\frac{1}{z\times\overline{z+1}}$. Quare habitâ ratione numeri 2, quem ex numeratoribus rejecimus, ſumma totius Seriei, à termino in quo eſt z in infinitum continuatæ, exhibetur per formulam $\frac{1}{2z\times\overline{z+1}}$; adeoque ſumma Seriei integræ eſt $\frac{1}{2\times1\times2}=\frac{1}{4}$

In Serie *B* ſunt $n=1$, $p=5$, $q=3$, primo valore z exiſtente 1. Eſt itaque $N=\overline{z+4}\times\overline{z+3}\times\overline{z+2}-z\times\overline{z+1}\times\overline{z+2}=6\times\overline{z+2}|^2$. Ipſius autem $z+2$ valores continui ſunt 3, 6, 9, *&c.* qui quoniam omnes ſunt diviſibiles per 3, ponendo $z+2=3x$, fit $N=6\times\overline{3x}|^2=6\times9x^2=54x^2$, ipſius x valoribus continuis exiſtentibus 1, 2, 3, *&e.* Rejecto itaque numero dato 54, hinc prodeunt numeratores 1, 2^2, 3^2, *&c.* hoc eſt 1, 4, 9, *&c.* Formula etiam Integralis eſt $\frac{1}{z\times\overline{z+1}}$; quare habitâ ratione numeri 54 quem ex numeratoribus rejecimus, ſumma Seriei à termino in quo eſt z in infinitum continuatæ eſt $\frac{1}{54z\times\overline{z+1}}$. Unde ſumma Seriei integræ eſt $\frac{1}{108}$.

In Serie denique *C* ſunt $n=1$, $p=5$ $q=4$, & primus valor $z=1$. Unde fit $N=\overline{z+4}\times\overline{z+3}\times\overline{z+2}\times\overline{z+1}-z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}=4\times\overline{z+1}$

×

$\overline{z+2} \times \overline{z+3}$. Valores autem *N* per hanc formulam prodeuntes ſemper poſſunt dividi per $4 \times 2 \times 3 \times 4 = 96$. Ergò hoc diviſore rejecto prodeunt numeratores 1, 14, 55, 140, *&c.* Et formula Summæ, habitâ ratione numeri 96, eſt $\frac{1}{96z}$. Adeoque Summa Seriei integræ eſt $\frac{1}{96}$.

Scholium 1. Per Propoſitiones has duas noviſſimas nullo negotio inveniri poſſunt Series quot libuerit ſummabiles. Et viciſſim oblatâ Serie hujus ſpeciei, ſi ſummari poteſt, ejus ſumma plerumque revocatur ad alterutram ex his Propoſitionibus. In examine tamen ſolertiâ eſt opus. Optime autem procedit ſi termini Seriei oblatæ revocentur ad formulam *Prop.* III. Sic *e. gr.* propoſitâ Serie $\frac{7}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} + \frac{11}{7 \cdot 9 \cdot 11 \cdot 13 \cdot 15} + \frac{15}{11 \cdot 13 \cdot 15 \cdot 17 \cdot 19} +$ *&c.* Denominatores ſic ſcribi poſſunt $3 \cdot 7 \cdot 11 \times 5 \cdot 9$, $7 \cdot 11 \cdot 15 \times 9 \cdot 13$, $11 \cdot 15 \cdot 19 \times 13 \cdot 17 \cdot$ *&c.*

Unde juxta *Prop.* III. fit $n = 4$, $m = 4$, $p = 2$, $q = 1$, primus valor $z = 3$, primus valor $u = 5$. Hinc formula Numeratoris invenitur $4 \times \overline{z + 2u + 8}$, Eſt autem $z + 2u + 8$ ſemper diviſibile per 3; quare rejectis diviſoribus datis 4 & 3, per hanc formulam prodeunt Numeratores 7, 11, 15, *&c.* iidem ac Numeratores in Serie propoſita, quæ proinde ſummabitur per illam propoſitionem.

2. Cùm Series illas *A*, *B*, *C*, communicaveram cum D. *Taylor*, reſcripſit ſe earum ſummas inveniſſe primam quidem *A* & tertiam *C*, eas revocando ad caſus ſimplices Methodi Incrementorum, tertiam *C*, *e g.* revocavit ad hanc formam $\frac{1}{24} \times \overline{\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \frac{1}{13 \cdot 17}\ \&c.}$, ut habeatur ſumma per præcepta tradita in *Scholio Prop.* 1.

In Serie autem ſecundâ *B*, cùm hoc non æquè ſucceſſit, ſequenti uſus eſt Analyſi, quam, ipſius venia jam impetratâ, ob ejus eximiam elegantiam huc transferre non piget. “ Seriei iſtius terminus [in Stylo ejus] ex“ hibetur per formulam $\frac{\overline{z+2}\times\underset{\cdot}{z}}{27\,z\times\overline{z+1}\times z_{,}\times\overline{z_{,}+1}}$; pro $z+3$ in denominatore ſcripto $z_{,}$, quoniam eſt $\underset{\cdot}{z}=3$.
“ Pone $\frac{B}{27\,C}$ æquale eſſe Integrali quæſito, hoc eſt $\frac{B}{C}$
“ eſſe Integrale ipſius $\frac{\overline{z+2}\times\underset{\cdot}{z}}{z\cdot\overline{z+1}\times z_{,}\cdot\overline{z_{,}+1}}$, ſepoſito divi“ ſore dato 27. Ipſius autem $\frac{B}{C}$ incrementum eſt
“ $\frac{\underset{\cdot}{B}C-B\underset{\cdot}{C}}{CC_{,}}$. Debet ergo $\frac{\underset{\cdot}{B}C-B\underset{\cdot}{C}}{CC_{,}}$ idem eſſe ac $\frac{\overline{z+2}\times\underset{\cdot}{z}}{z\cdot\overline{z+1}\times z_{,}\cdot\overline{z_{,}+1}}$. Comparando denominatores inveni“ tur $C=z\times\overline{z+1}$. Hinc itaque ſumendo incremen“ ta fit $\underset{\cdot}{C}=2z\underset{\cdot}{z}+\underset{\cdot}{z}^{2}+\underset{\cdot}{z}$ ($=2\underset{\cdot}{z}z+4z$, quoniam
“ eſt $\underset{\cdot}{z}=3$,) His valoribus in locum C & $\underset{\cdot}{C}$ ſubſtitu“ tis prodit $\underset{\cdot}{B}C-B\underset{\cdot}{C}=\overline{zz+z}\,\underset{\cdot}{B}-2\underset{\cdot}{z}\times\overline{z+2}\,B$,
“ quod debet eſſe idem ac $\overline{z+2}\times\underset{\cdot}{z}$. Sit $B=a+v$,
“ exiſtente *a* ipſius *B* parte invariabili, & *v* parte va“ riabili. Tum ſumendo incrementa fit $\underset{\cdot}{B}=\underset{\cdot}{v}$. Unde
“ ad invenienda *a* & *v* habetur æquatio $\overline{zz+z}\,\underset{\cdot}{v}$
“ $-2\underset{\cdot}{z}\times\overline{z+2}\times\overline{a+v}=\overline{z+2}\times\underset{\cdot}{z}$, quæ ſic ſcribi
“ poteſt $\overline{zz+z}\,\underset{\cdot}{v}-2\underset{\cdot}{z}\times\overline{z+2}\,v=\underset{\cdot}{z}\times\overline{z+2}\times\overline{1+2a}$
“ vel etiam $C\underset{\cdot}{v}-\underset{\cdot}{C}v=\underset{\cdot}{z}\times\overline{z+2}\times\overline{1+2a}$. Pone
“ $1+2a=0$ (unde fit $a=-\frac{1}{2}$,) & fit $C\underset{\cdot}{v}-\underset{\cdot}{C}v=0$;

“ ubi fieri poteſt $\underset{\cdot}{v} = 0$, (quoniam æquationis termini
“ ſinguli afficiuntur vel ab v, vel ab $\underset{\cdot}{v}$) Hinc ergò fit $B =$
“ $a = \frac{-1}{2}$, adeoque $\frac{B}{C} = \frac{-1}{2z \times \overline{z+1}}$. Unde habitâ ratione
“ diviſoris 27, Integrale quæſitum fit $\frac{-1}{54 \times z \times \overline{z+1}}$.
“ Sed & comparando æquationem $C\underset{\cdot}{v} - \underset{\cdot}{C}v = 0$ cum
“ formulâ generali $\frac{\underset{\cdot}{B}C - B\underset{\cdot}{C}}{CC} = 0$, inde etiam concludere
“ licet eſſe $\frac{v}{C} =$ quantitati datæ, (quoniam ipſius
“ incrementum eſt 0.) Unde pro n ſumpto quovis
“ numero dato, fit $v = nC$, atque $B = -\frac{1}{2} + nC$.
“ Quo pacto Integrale quæſitum fit $\frac{B}{C} = \frac{-\frac{1}{2} + nC}{C} = \frac{-1}{2C}$
“ $+ n$, quod ab Integrali prius invento differt quantitate
“ datâ n. Hoc inde fit, quòd, ut in quadraturâ
“ Curvarum Area inventa augeri poteſt vel minui areâ
“ datâ, ſic in Methodo incrementorum Integrale inventum
“ augeri poteſt vel minui quantitate datâ. Per
“ Integrale autem primum, ubi deeſt n, exhibetur
“ ſumma Seriei in infinitum continuatæ.

Prop. V.

Creſcente z per unitates, & exiſtentibus a, b, c, *&c.* numeris datis integris, quorum nullæ inter ſe æquantur; invenire Integrale ipſius $\frac{1}{z \times \overline{z+a} \times \overline{z+b} \times \overline{z+c} \times \&c.}$.

Solutio. Ducendo tam numeratorem quam denominatorem fractionis in terminos $z+1$, $z+2$, *&c.* $z+a+1$, $z+a+2$, *&c.* $z+b+1$, $z+b+2$, *&c.* $z+c+1$, $z+c+2$, *&c.* in denominatore deficientes, revocetur Denominator ad formulam $z \times \overline{z+1}$

$\times\overline{z+2}\times$ &c. denominatoris in *Prop.* I. *Schol. n.* 3. Deinde revocetur Numerator ad formam $A + Bz + Cz \times\overline{z+1} + Dz\times\overline{z+1}\times\overline{z+2} +$ &c. Tum applicando terminos ad Denominatorem novum $z\times\overline{z+1}\times\overline{z+2}\times$ &c. revocetur fractio ad hanc formam

$$\frac{A}{z\times\overline{z+1}\times\&c.} + \frac{B}{\overline{z+1}\times\overline{z+2}\times\&c.} + \frac{C}{\overline{z+2}\times\overline{z+3}\times\&c.} + \frac{D}{\overline{z+3}\times\overline{z+4}\times\&c.}\ \&c.$$

Unde denique quæratur Integrale per *Schol. Prop.* I. *n.* 3.

Ratio Solutionis per ſe ſatis eſt manifeſta.

Scholium 1. Hujus Solutionis tota difficultas latet in revocatione numeratoris ad formam requiſitam, quod tamen quomodo ſit faciendum uno exemplo patebit. Proponatur itaque factum $\overline{z+2}\times\overline{z+3}\times\overline{z+7}$, quod ad formam propoſitam ſit revocandum. Terminos itaque evolvo gradatim ut ſequitur. Factorem primum $z+2$ ſic ſcribo $2+z$, cujus terminum primum 2 duco in $3+z$, unde fit $6+2z$: Terminum ſecundum z duco in $2+\overline{z+1}$ $(=z+3)$ unde fit $2z+z\times\overline{z+1}$. Dein facta in unam ſummam colligendo, fit $\overline{z+2}\times\overline{z+3} = \begin{matrix}6+2z\\+2z\end{matrix} + z\times\overline{z+1} = 6+4z+z\times\overline{z+1}$. Supereſt ut hoc ducatur in $z+7$. Itaque terminum primum 6 duco in $7+z$ $(=z+7)$ unde fit $42+6z$; terminum ſecundum $4z$ duco in $6+\overline{z+1}$ $(=z+7)$ unde fit $24z+4z\times\overline{z+1}$; terminum tertium $z\times\overline{z+1}$ duco in $5+\overline{z+2}$ $(=z+7,)$ unde fit $5z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Factis itaque in unum collectis ut prius, fit $\overline{z+2}\times\overline{z+3}\times\overline{z+4} = 42+30z+9z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Et ad eundem modum procedere licet in aliis caſibus.

2. Sit autem exemplum Propoſitionis in fractione $\frac{1}{z\times\overline{z+2}\times\overline{z+5}}$. Reſtituendo factores $z+1$, $z+3$, $z+4$ in Denominatore deficientes, fractio fit $\frac{\overline{z+1}\times\overline{z+3}\times\overline{z+4}}{z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$. Revocandus itaque eſt Numerator $\overline{z+1}\times\overline{z+3}\times\overline{z+4}$ ad formam requiſitam. Itaque per methodum jam traditam fit primo $\overline{z+1}\times\overline{z+3}=1\times\overline{3+z}+z\times\overline{2+z+1}$ $=3+z+2z+z\times\overline{z+1}=3+3z+z\times\overline{z+1}$. Deinde $\overline{z+1}\times\overline{z+3}\times\overline{z+4}=3\times\overline{4+z}+3z\times\overline{3+z+1}+z\times\overline{z+1}\times\overline{2+z+2}=12+3z+9z$ $+3z\times\overline{z+1}+2z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$ $=12+12z+5z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Applicando hoc factum ad Denominatorem $z\times\overline{z+1}\times$ &c. $\times\overline{z+5}$ fractio tandem revocatur ad hanc formam

$$\frac{12}{z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}+\frac{12}{\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}+\frac{5}{\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}+\frac{1}{\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$$

Cujus denique Integrale eſt

$$\frac{-12}{5z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-12}{4\cdot\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-5}{3\cdot\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-1}{2\cdot\overline{z+3}\times\overline{z+4}}.$$

3. Quando duo tantùm ſunt factores z & $z+a$, exhibebitur etiam Integrale per formulam $\frac{1}{z}-\frac{1-a}{2z\times\overline{z+1}}$ $-\frac{\overline{1-a}\times\overline{2-a}}{3z\times\overline{z+1}\times\overline{z+2}}-\frac{\overline{1-a}\times\overline{2-a}\times\overline{3-a}}{4z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}$ &c. Seriem nempe continuando donec abrumpatur per evaneſcentiam

neſcentiam terminorum. Si Factores duo ſint z & $z-a$ exhibebitur Integrale per formulam $\frac{-1}{z-1}-\frac{\overline{-1+a}}{2\cdot\overline{z-1}\cdot\overline{z-2}}-\frac{\overline{-1+a}\times\overline{-2+a}}{3\cdot\overline{z-1}\cdot\overline{z-2}\cdot\overline{z-3}}-$ &c. Poteſt idem Integrale exprimi utroque modo, prout fractionis oblatæ factor vel minor vel major ſumatur pro z.

4. Si primus valor z ſit $a+1$, migrabit formula poſterior in hanc $\frac{-1}{a}\times\frac{1}{1}\times\frac{1}{2}\times\frac{1}{3}+$ &c. uſque $\frac{1}{a}$ incluſivè, quâ, cum ſigno contrario, exhibetur ſumma Seriei $\frac{1}{1\times\overline{1+a}}+\frac{1}{2\times\overline{2+a}}+\frac{1}{3\times\overline{3+a}}+$ &c. in infinitum continuatæ. Sit *e. gr.* $a=1$, atque Series erit $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+$ &c. $=\frac{1}{1}\times\frac{1}{1}=1$. Si $a=2$, erit Series $\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+$ &c. $=\frac{1}{2}\times\overline{\frac{1}{1}+\frac{1}{2}}=\frac{3}{4}$; Si $a=3$, Series erit $\frac{1}{1\times4}+\frac{1}{2\times5}+\frac{1}{3\times6}+\frac{1}{4\times7}$ &c. $=\frac{1}{3}\times\frac{1}{1}+\frac{1}{2}+\frac{1}{3}=\frac{11}{18}$.

5. Ex eâdem Serie $\frac{1}{1\times\overline{1+a}}+\frac{1}{2\times\overline{2+a}}+\frac{1}{3\times\overline{3+a}}+$ &c. pro diverſo valore a oriuntur Series plures formâ ſatis elegantes, quarum nonnullas Lectori ob oculos ſiſtere, credo, ingratum non erit.

Si pro a ſumantur ſucceſſivè numeri pares, 2, 4, 6, 8, &c. Series erunt

Si $a=2$) $\frac{1}{1\times\overline{1+2}}+\frac{1}{2\times\overline{2+2}}+\frac{1}{3\times\overline{3+2}}+\frac{1}{4\times\overline{4+2}}+$ &c.

4) $\frac{1}{1\times\overline{1+4}}+\frac{1}{2\times\overline{2+4}}+\frac{1}{3\times\overline{3+4}}+\frac{1}{4\times\overline{4+4}}+$ &c.

6) $\frac{1}{1\times\overline{1+6}}+\frac{1}{2\times\overline{2+6}}+\frac{1}{3\times\overline{3+6}}+\frac{1}{4\times\overline{4+6}}+$ &c.

8) $\frac{1}{1\times\overline{1+8}}+\frac{1}{2\times\overline{2+8}}+\frac{1}{3\times\overline{3+8}}+\frac{1}{4\times\overline{4+8}}+$ &c.

$$\text{Vel } \frac{1}{4-1}+\frac{1}{9-1}+\frac{1}{16-1}+\frac{1}{25-1}+\&c.$$

$$\frac{1}{9-4}+\frac{1}{16-1}+\frac{1}{25-4}+\frac{1}{36-4}+\&c.$$

$$\frac{1}{16-9}+\frac{1}{25-9}+\frac{1}{36-9}+\frac{1}{49-9}+\&c.$$

$$\frac{1}{25-16}+\frac{1}{36-16}+\frac{1}{49-16}+\frac{1}{64-16}+\&c.$$

$$\text{Vel } \frac{1}{4-1}+\frac{1}{9-1}+\frac{1}{16-1}+\frac{1}{25-1}+\&c.$$

$$\frac{1}{4+1}+\frac{1}{9+3}+\frac{1}{16+5}+\frac{1}{25+7}+\&c.$$

$$\frac{1}{4+3}+\frac{1}{9+7}+\frac{1}{16+11}+\frac{1}{25+15}+\&c.$$

$$\frac{1}{4+5}+\frac{1}{9+11}+\frac{1}{16+17}+\frac{1}{25+23}+\&c.$$

Si pro *a* ſumantur ſucceſſive numeri impares 1, 3, 5, 7, &c. Series erunt

$$=1)\ \frac{1}{1\times\overline{1+1}}+\frac{1}{2\times\overline{2+1}}+\frac{1}{3\times\overline{3+1}}+\frac{1}{4\times\overline{4+1}}+\&c.$$

$$3)\ \frac{1}{1\times\overline{1+3}}+\frac{1}{2\times\overline{2+3}}+\frac{1}{3\times\overline{3+3}}+\frac{1}{4\times\overline{4+3}}+\&c.$$

$$5)\ \frac{1}{1\times\overline{1+5}}+\frac{1}{2\times\overline{2+5}}+\frac{1}{3\times\overline{3+5}}+\frac{1}{4\times\overline{4+5}}+\&c.$$

$$7)\ \frac{1}{1\times\overline{1+7}}+\frac{1}{2\times\overline{2+7}}+\frac{1}{3\times\overline{3+7}}+\frac{1}{4\times\overline{4+7}}+\&c.$$

$$\text{Vel } \frac{1}{2}\times\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\&c.$$

$$\frac{1}{2}\times\frac{1}{3-1}+\frac{1}{6-1}+\frac{1}{10-1}+\frac{1}{15-1}+\&c.$$

$$\frac{1}{2}\times\frac{1}{6-3}+\frac{1}{10-3}+\frac{1}{15-3}+\frac{1}{21-3}+\&c.$$

$$\frac{1}{2}\times\frac{1}{10-6}+\frac{1}{15-6}+\frac{1}{21-6}+\frac{1}{28-6}+\&c.$$

$$\text{Vel } \frac{1}{2}\times\frac{1}{1+0}+\frac{1}{3+0}+\frac{1}{6+0}+\frac{1}{10+0}+\&c.$$

$$\frac{1}{2}\times\frac{1}{1+1}+\frac{1}{3+2}+\frac{1}{6+3}+\frac{1}{10+4}+\&c.$$

$$\frac{1}{2}\times\frac{1}{1+2}+\frac{1}{3+4}+\frac{1}{6+6}+\frac{1}{10+8}+\&c.$$

$$\frac{1}{2}\times\frac{1}{1+3}+\frac{1}{3+6}+\frac{1}{6+9}+\frac{1}{10+12}+\&c.$$

6. Ante aliquot annos D. *Jac. Bernoulli* Geometra inſignis invenit ſummam Seriei cujuſlibet, cujus Numeratores conſtituunt Seriem æqualium, Denominatores verò conſtituunt, vel Seriem quadratorum dato aliquo quadrato *Q* minutorum, vel Seriem Triangulorum, dato aliquo Triangulo *T* minutorum. Hæc invenit ille obſervando quod hujuſmodi Series oriantur ex ablatione Seriei Harmonicè proportionalium truncatæ ab eâdem Serie integrâ; nempe ita ut numerus terminorum deficientium in Serie truncata, ſit, vel duplus lateris dati quadrati *Q*, vel duplus unitate auctus lateris dati Trianguli *T*. Idem etiam obſervavit fruſtrà quæri ſummam Seriei reciprocæ Quadratorum. Hoc idem etiam verum eſt de reciprocis Cuborum, vel aliarum quarumlibet dignitatum numerorum in progreſſione Arithmeticâ. Ratio eſt, quòd nulla intercedit differentia inter factores denominatorum, quod ad hujuſmodi ſummationes ſemper requiri conſtat ex Methodo ſumendi differentias in *Scholio Prop.* I. jam explicatâ. Nam ſi per formulam aliquam exhiberi poſſet ſumma quæſita, differentia iſtius formulæ exhiberet terminos Seriei propoſitæ: ſed in tali differentiâ denominator ſemper afficitur per factores ab invicem diverſos, quod quoniam in Seriebus prædictis non obtinet, ſummæ Serierum hujuſmodi in terminis finitis haberi nequeunt. Ad eundem ferè modum, argumento petito à *Prop.* III. & IV. demonſtrari poteſt ſummas Serierum exhiberi non poſſe in terminis numero finitis, quarum Numeratores conſtituunt Seriem æqualium. Denominatores vero conſtant ex certo numero terminorum in progreſſione Arithmeticâ, maximo factore cujuſvis termini minore exiſtente quàm factor minimus in termino proxime inſequenti, cujuſmodi eſt Series $\frac{1}{1 . 2}+\frac{1}{3 . 4}+\frac{1}{5 . 6}+$ &c.

7. Jam liceret regulas nonnullas tradere quas pro caſibus quibuſdam ſingularibus concinnavi; ſed hæc

nos longius abducerent. Sufficiat itaque quæ generaliora ſunt explicaſſe, & ſimul monuiſſe, ad novæ hujuſce Serierum infinitarum doctrinæ provectionem nihil magis facere, quam ſi excogitentur formulæ generaliores ſummarum, ex quarum differentiis, per regulas ſupra traditas computatis, deinde conficiantur Canones quantitatum ſummabilium; ita ferè ut jam factum eſt in Calculo Integrali, *h. e.* in Stylo *Newtoniano*, in Methodo Fluxionum.

8. Reſtituendo factores in Denominatore deficientes potuiſſet præſens Problema revocari ad *Propoſitionem* II. Sed & in terminis generalioribus proponi poteſt, nempe pro Numeratore ſumptâ quâvis Formulâ, cujus differentia aliqua datur. Sub eâ tamen conditione ut dimenſiones Denominatoris ad minimum binario ſuperent Dimenſiones Numeratoris; aliàs enim ſumma Seriei in terminis numero finitis haberi nequit. Sit hujus rei exemplum in Serie $\frac{1}{1.3.5.7}+\frac{4}{2.4.6.8}+\frac{9}{3.5.7.9}+\frac{16}{4.6.8.10}+$ &c. ubi Numeratores ſunt numerorum naturalium quadrata. Applicando tum Numeratores tum Denominatores ad numeros naturales, Series revocatur ad formam ſimpliciorem $\frac{1}{3.5.7}+\frac{2}{4.6.8}+\frac{3}{5.7.9}+\frac{4}{6.8.10}+$ &c. Per p deſignatis numeris naturalibus 1, 2, 3, 4, &c. terminus Seriei deſignabitur per formulam $\frac{p}{\overline{p+2}\times\overline{p+4}\times\overline{p+6}}$; vel per formulam $\frac{z-2}{z\times\overline{z+2}\times\overline{z+4}}$, nempe pro $p+2$ ſcripto z. Quoniam progrediendo de termino in terminum augetur z per unitates, reſtituendi ſunt factores in denominatore deficientes $z+1$, $z+3$, & hoc pacto revocatur terminus Seriei ad formulam $\frac{\overline{z-2}\times\overline{z+1}\times\overline{z+3}}{z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}$. Per methodum in hâc Propoſitione jam explicatam revocatur

vocatur numerator ad formam $-6-6z-z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Unde habita ratione denominatoris Terminus revocatur ad formam $\frac{-6}{z\times\overline{z+1}\times\&c\times\overline{z+4}}+\frac{-6}{\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-1}{\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{1}{\overline{z+3}\times\overline{z+4}}$. Adeoque ſumendo Integrale fit $\frac{6}{4z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}+\frac{6}{3\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}+\frac{1}{2\times\overline{z+2}\times\overline{z+3}}+\frac{-1}{z+3}$; quo, ſub ſigno contrario, exhibetur ſumma Seriei in infinitum continuatæ, incipientis à termino $\frac{z-2}{z\times\overline{z+2}\times\overline{z+4}}$. Summa itaque Seriei integræ incipientis à termino $\frac{1}{3\cdot5\cdot7}$ eſt $\frac{31}{240}$.

Si per *Prop.* II. procedere eſſet animus, ex formulâ $z-2\times\overline{z+1}\times\overline{z+3}$ collectis numeratoribus primis 24, 70, 144, 252, ſumendo eorum differentias haberentur $46=b$, $28=c$, $6=d$, $e=0=$ &c. exiſtente $M=24$; unde per *Lem.* 2. prodiret formula $-6-6z-z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$, quâ deſignatur Terminus, eadem ac ſupra; atque pergendo per *Prop.* II. haberetur ſumma.

Prop. VI. *Prob.*

Invenire ſummam quotlibet terminorum Seriei Fractionum, quarum Numeratores & Denominatores conſtituunt lineas duas quaſvis tranſverſas in Triangulo Arithmetico *Paſchalii*; nempe cujus generatores ſunt unitates.

Solutio. Per n deſignetur Ordo Seriei Numeratorum in Triangulo Arithmetico, & ſit p differentia inter ordinem Numeratorum & Denominatorum, & per q deſignetur numerus terminorum quorum ſumma requiritur. Tum ſi Denominatores ſint plurium dimenſionum quàm ſunt Numeratores, Summa exhibebitur per formulam primam ſequentem; ſi dimenſiones Numeratorum plures ſint quàm dimenſiones Denominatorum, Summa exhibebitur per formulam ſecundam.

Formula I.

$$\frac{n+p-1}{p-1} - \frac{n.\overline{n+1}.\overline{n+2}.\&c.\overline{n+p-1}}{\overline{p-1}\times\overline{n+q}.\overline{n+q+1}.\&c.\overline{n+q+p-2}}$$

Formula II.

$$-\frac{n-p-1}{p+1} + \frac{\overline{q+n-1}.\overline{q+n-2}.\&c.\overline{q+n-p-1}}{\overline{p+1}\times\overline{n-1}.\overline{n-2}.\&c.\overline{n-p}}$$

Ex. 1. Inveniendum ſit aggregatum ſex primorum terminorum Seriei $\frac{1}{1}+\frac{4}{7}+\frac{10}{28}+\frac{20}{84}+\frac{35}{210}+\frac{56}{462}+$ *&c.* ubi Numeratores conſtituunt lineam quartam, Denominatores conſtituunt lineam ſeptimam in Triangulo Arithmetico Sunt itaque $n=4$, $p=3$, $q=6$; & quoniam dimenſiones Denominatorum ſuperant dimenſiones Numeratorum, dabitur ſumma per Formulam primam; nempe $\frac{4+3-1}{3-1} - \frac{4.5.6}{\overline{3-1}\times\overline{4+6}\times\overline{4+7}}$ ſive $3-\frac{6}{11}=2\frac{5}{11}$.

Ex. 2. Quæratur ſumma ſex primorum terminorum Seriei $\frac{1}{1}+\frac{7}{4}+\frac{28}{10}+\frac{84}{20}+\frac{210}{35}+\frac{462}{56}+$ *&c.* cujus termini ſunt terminorum Seriei prioris reciproci. Sunt itaque $n=7$, $p=3$, $q=6$, adeoque per formulam ſecundam ſumma fit $-\frac{3}{4}+\frac{12.11.10.9}{4\times 6.5.4}=24$.

Scholium 1. Formulas in hac propoſitione exhibitas ante biennium communicavi cum Viris celeberrimis *Moivreo* & *Bernoulliis.* Facile autem derivari poſſunt ex præceptis in *Prop.* I. traditis. Sit exemplum in Serie priori $\frac{1}{1}+\frac{4}{7}+\frac{10}{28}+$ *&c.* Per p deſignato loco

Termini in Serie hâc, exhibetur Terminus per formulam $\frac{4\cdot5\cdot6}{\overline{p+3}\cdot\overline{p+4}\cdot\overline{p+5}}$; Unde regrediendo ad Integrale, ſumma Seriei incipientis à termino illo exhibetur per formulam $\frac{4\cdot5\cdot6}{2\times\overline{p+3}\times\overline{p+4}}$; adeoque pro p ſumpto 1, Series integra fit $\frac{4\cdot5\cdot6}{2\cdot4\cdot5}=3$, atque ſumma primorum ſex terminorum fit $3-\frac{4\cdot5\cdot6}{2\cdot10\cdot11}$, omninò ut per formulam jam exhibetur.

2. In formulâ primâ ſumma Seriei in infinitum continuatæ eſt $\frac{n+p-1}{p-1}$, evaneſcente jam parte alterâ formulæ. Sed in caſu formulæ ſecundæ ſumma hæc eſt infinitum quid, cujus ſpecies, reſpectu numeri infiniti q, exhibetur per formulæ partem alteram, quæ in hoc caſu fit $\frac{q^{p+1}}{\overline{p+1}\times\overline{n-1}\cdot\overline{n-2}\cdot\&c.\overline{n-p}}$.

3. De hujuſmodi Seriebus in epiſtolâ datâ menſe *Maio* 1716, ſic ad me ſcripſit Vir. Ill. *D. Leibnitius*, quem magno Scientiarum damno nobis nuper ereptum lugemus. “ Il me ſemble qu'autrefois j'ay auſſi ſommé “ quelques Series ou ſuittes comme $\frac{1}{1}+\frac{2}{4}+\frac{3}{10}+\frac{4}{20}$ “ $+\frac{5}{35}+\frac{6}{56}+$ *&c.* Le terme de cette ſuitte exprimé “ Analytiquement eſt $\frac{x}{x\cdot\overline{x+1}\cdot\overline{x+2}\times\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}}$ “ $=\frac{1\cdot2\cdot3}{\overline{x+1}\cdot\overline{x+2}}=\frac{6}{xx+3x+2}$. On demande donc “ la ſomme d'une ſuitte donneé, dont un terme ſoit “ $\frac{ll}{xx+3lx+2ll}$ ou x ſignifie les nombres naturales “ 1, 2, 3, 4, *&c.* & l ſignifie l'Unité, ou la difference “ des x. Suppoſons que le terme de la ſuitte ſom-

“ matrice demandée ſoit $\frac{fx}{mx+nl}=\frac{\odot}{\text{☽}}$. Or Diff. $\frac{\odot}{\text{☽}}=$
“ $-\frac{\odot}{\text{☽}}+\frac{\odot+d\odot}{\text{☽}+d\text{☽}}=\frac{\text{☽}\,d\odot-\odot\,d\text{☽}}{\text{☽}\text{☽}+\text{☽}\,d\text{☽}}$: ſed $d\odot=fdx$,
“ & $d\text{☽}=mdx=ml$; donc la Difference de $\frac{\odot}{\text{☽}}$ eſt $=$
“ $\frac{nfll}{\begin{matrix}mmxx+2mnlx+nnll\\ +mmlx+mnll\end{matrix}}$. Maintenant il faut faire
“ $\frac{nfll}{\begin{matrix}mmxx+2mnlx+nnll\\ +mmlx+mnll\end{matrix}}=\frac{mfll}{mmxx+3mmlx+2mmll}$
“ c'eſt a dire, il faut identifier ces deux formules, ou la
“ donnée eſt Multipliée per $\frac{nf}{mm}$: donc égalant les
“ termes reſpectifs, puiſque les xx conviennent, on
“ aura par les x, $2n+m=3m$, c'eſt adire il y aura
“ $m=n$, & par les abſolus on aura $nn+mn=2mm$,
“ ce qui donne encore $m=n$; donc l'identification
“ reuſſir, & nous pouvons faire $n=m=l=1$, &
“ $f=1$ (car f demeure arbitraire) & le terme de la
“ ſuitte ſommatrice ſera $\frac{x}{x+1}$, car diff. $\frac{x}{x+1}$ donne
“ $-\frac{x}{x+1}+\frac{x+1}{x+2}=\frac{1}{xx+3x+2}$, & par conſequente
“ $\frac{6x}{x+1}$ donne la ſomme des $\frac{x}{x.\overline{x+1}.\overline{x+2}\times\frac{1}{1}.\frac{1}{2}.\frac{1}{3}}$
“ $3,\ 4,\ \frac{9}{2},\ \frac{24}{5},\ 5,\ \frac{36}{7}$, *&c. Series ſummatrix, cujus ter-*
“ *minus* $\frac{6x}{x+1}$.
“ $\frac{1}{1}+\frac{2}{4}+\frac{3}{10}+\frac{4}{20}+\frac{5}{35}+$ *&c. Series ſummanda, cu-*
“ *jus terminus* $\frac{x}{x.\overline{x+1}.\overline{x+2}\times\frac{1}{1}\times.\frac{1}{3}}$ Et pour
“ ſ'en ſervir aux ſommations, les 5 termes, *par Ex.* de

" la ſuitte donnée feront $\frac{36}{7} - 3 = \frac{15}{7}$. Et generalle-
" ment la ſomme des termes juſqu'a quelque terme
" $\frac{x}{x \,.\, \overline{x+1}\ \overline{x+2} \times \frac{1}{1} \,.\, \frac{1}{2} \,.\, \frac{1}{3}}$ excluſivement, ſera $\frac{6x}{x+1}$
" -3 : Et pour la ſomme de la ſuitte entiere a l'infi-
" nie, x devient infini, & $\frac{6x}{x+1} = 6$: donc la ſomme
" de toute la ſuitte eſt $6 - 3 = 3$, comme vous
" l'avez trouvé.

" Cette methode eſt le calcul des differences ap-
" pliqué aux Nombres ; & il faut vous avouer qu'a-
" vant que de l'appliquer aux Figures, & même avant
" que d'avoir été Geometre, Je le prattiquai en quel-
" que façon dans les nombres ; ayant trouvé encore
" jeune garçon que les ſuittes dont les Numerateurs
" fuſſent des Unites, & dont les Denominateurs fuſſent
" les Nombres figurés, comme Triangulaires Pyrami-
" daux &c. etoient les differences 1$^{\text{eres}}$, 2$^{\text{es}}$, 3$^{\text{emes}}$, &c.
" multipliées par les conſtantes de la ſuitte $\frac{1}{1} + \frac{1}{2} + \frac{1}{3}$
" $+ \frac{1}{4} +$ &c. & par conſequent ſommables. Mais
" quand je devins un peu Geometre & Analyſte, Je
" vis qu'il y avoit moyen de venir a bout de telles
" ſommations par une Methode generalle, autant qu'il
" etoit poſſible ; & que le calcul des differences eſtoit
" encore plus commode dans la Geometrie que dans
" les Nombres, puis qu'il y a plus d'evanouiſſements,
" & que les differences repondent aux Tangentes, les
" ſommes aux Quadratures. Cette methode generalle
" de chercher la ſuitte ſommatrice de la ſuitte donnée,
" quand elle eſt poſſible, reuſſit toujours, quand le terme
" de la ſuitte donnée exprimé Analytiquement n'a
" point la quantité variable enveloppé dans une racine,
" ny entrant dans l'expoſant ; & alors, on peut tou-
" jours

“ jours determiner la ſuitte ſommatrice, ou prouver
“ qu'il eſt impoſſible d'en trouver. Et la choſe reuſſit
“ méme bien ſouvent, lors méme que la variable en-
“ tre dans l'Expoſant. Mais comme il y a quelque-
“ fois des Quadratures particulieres de quelques por-
“ tions d'une Figure, dont ou ne ſçauroit donner la
“ Quadrature generalle ou la Figure quadratrice; de
“ méme on peut trouver quelquefois la ſomme de
“ toute la ſuitte, ou d'un certaine partie, quoy qu'on
“ ne puiſſe pas trouver la ſomme de chaque partie; &
“ alors il faut avoir recours a des Methodes particulieres,
“ dont on n'eſt pas toujours le maiſtre, noſtre Analyſe
“ n'eſtant pas encore portée a ſa perfection.

Prop. VII. *Prob.*

Invenire ſummam Seriei cujus Numeratores conſtituunt lineam quamlibet erectam in Triangulo Arithmetico *Paſchalii*, Denominatores vero conſtituunt lineam quamlibet tranſverſam.

Solutio. Deſignetur ordo lineæ erectæ per p, ordo lineæ tranſverſæ per q, & ſit m aggregatum tot terminorum primorum in lineâ erectâ ordinis $p+q-1$ quot ſunt unitates in $q-1$, atque ſumma quæſita erit $\overline{2^{p+q-2}-m}\times\frac{1 . 2 . 3 . \&c. \overline{q-1}}{p . \overline{p+1} . \&c. \overline{p+q-2}}$.

Ex. 1. Proponatur Series $\frac{1}{1}+\frac{5}{4}+\frac{10}{10}+\frac{10}{20}+\frac{5}{35}+\frac{1}{56}$ Ubi Numeratores conſtituunt lineam ſextam erectam, Denominatores occupant lineam quartam tranſverſam. In hoc itaque caſu ſunt $p=6$, $q=4$, $p+q-1=9$, $q-1=3$, adeoque $m=1+8+28=37$ *i e.* tribus terminis primis lineæ nonæ erectæ. Unde fit ſumma quæſita $\overline{2^{8}-37}\times\frac{1 . 2 . 3}{6 . 7 . 8}=\frac{219}{56}$.

Ex. 2. Conſtituant Numeratores lineam centeſimam erectam, & ſint Denominatores Numeri Trigonales, qui occupant lineam tertiam tranſverſam. Tum erunt

$p=100$, $q=3$, $m=102$ atque adeo ſumma quæſita fit $\overline{2^{101}-102} \times \frac{1 \cdot 2}{100 \cdot 101}$.

Cor. Si $q=2$, formula fit $\frac{2^p-1}{p}$, quâ exhibetur aggregatum primi termini, unâ cum ſemiſſe ſecundi, triente tertii, quadrante quarti, & ſic porrò, lineæ cujuſvis erectæ ordinis p Trianguli Arithmetici *Paſchalii*. Sic *v. gr.* eſt $\frac{1}{1}+\frac{5}{2}+\frac{10}{3}+\frac{10}{4}+\frac{5}{5}+\frac{1}{6}=\frac{2^6-1}{6}=10\frac{1}{2}$.

Prop. VIII. *Prob.*

Invenire ſummam ejuſdem Seriei, quando terminorum ſigna ſunt alternatim + & —.

Solutio. Summa quæſita exhibetur per formulam ſimpliciſſimam $\frac{q-1}{p+q-2}$.

Ex. Invenienda ſit ſumma Seriei $\frac{1}{1}-\frac{6}{9}+\frac{15}{45}-\frac{20}{165}+\frac{15}{495}-\frac{6}{1287}+\frac{1}{3003}$, ubi Numeratores conſtituunt lineam ſeptimam erectam, Denominatores conſtituunt nonam tranſverſam. In formulâ itaque pro p & q ſcriptis 7 & 9, fit ſumma $\frac{8}{14}$.

Manente eâdem Serie Numeratorum (nempe lineâ ſeptimâ erectâ) ſi pro Serie Denominatorum ſumantur ſucceſſivè lineæ tranſverſæ 2^{da}, 3^{tia}, 4^{ta}, *&c.* Summæ erunt $\frac{1}{7}$, $\frac{2}{8}$, $\frac{3}{9}$, $\frac{4}{10}$, $\frac{5}{11}$, *&c.* quæ ſic poſſunt ſcribi, $\frac{1}{7}$, $\frac{7}{28}$, $\frac{28}{84}$, $\frac{84}{210}$, $\frac{210}{462}$, *&c.* ubi tam Numeratores, quàm Denominatores excerpuntur ex lineâ tranſverſâ ordinis ſeptimi. Idem eveniret ſi loco ſeptimæ, Numeratores conſtituiſſent aliam quamlibet lineam erectam ordinis p; Summæ quippe orirentur ex applicatione terminorum lineæ tranſverſæ ejuſdem ordinis p ad terminos proximè ſequentes in eadem lineâ.

Propoſitiones hæ duæ noviſſimæ potius elegantes ſunt quàm utiles; quare Formularum noſtrarum demonſtrationem Lectoris ſolertia inveſtigandam relinquimus, ad Propoſitionem ultimam jam properantes, quæ tertiam continet Serierum ſpeciem, ob uſum multiplicem ſatis inſignem.

Lemma 5.

Sit Series quævis $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, *&c.* cujus terminorum Denominatores conſtituunt progreſſionem quamlibet Geometricam h, h^2, h^3, h^4, *&c.* Sint etiam Numeratorum primus A ($= M$), prima differentiarum primarum B, prima ſecundarum C, prima tertiarum D, quartarum E, & ſic porrò; & ſint $\frac{\alpha}{h}, \frac{\beta}{h^2}, \frac{\gamma}{h^3}, \frac{\delta}{h^4}$, *&c.* reſpectivè, aggregata, Unius, Duorum, Trium, Quatuor, vel plurium terminorum Seriei $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}$, *&c.* atque ſint Numeratorum primus a ($= \alpha$) prima differentiarum primarum b, prima ſecundarum c, prima tertiarum d, & ſic porrò: & ſit $h - 1 = q$. Tum ipſorum a, b, c, d, *&c.* valores erunt.

$$\begin{aligned} a &= A = \alpha = M \\ b &= hA + B \\ c &= qhA + hB + C \\ d &= q^2hA + qhB + hC + D \end{aligned}$$

& ſic porrò.

Demonſtratio.

Satis conſtat eſſe $a = \alpha = A = M$.

Termini $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, *&c.* Numeratoribus M, N, O, P,

&c. expreſſis per A, B, C, D, &c. transformantur in terminos $\frac{A}{h}$, $\frac{A+B}{h^2}$, $\frac{A+2B+C}{h^3}$, $\frac{A+3B+3C+D}{h^4}$ &c. Unde colligendo ſummas terminorum, inveniuntur Numeratores α, β, γ, δ, &c. nempe

$$\alpha = A$$

$$\beta = \overline{h+1}\,A + B$$

$$\gamma = \overline{h^2+h+1}\,A + \overline{h+2}\,B + C$$

$$\delta = \overline{h^3+h^2+h+1}\,A + \overline{h^2+2h+3}\,B + \overline{h+3}\,C + D$$

&c.

Unde ſumendo differentias fiunt

$$b = hA + B$$

$$c = qhA + hB + C$$

$$d = qqhA + qhB + hC + D$$

& ſic porrò, ut in Propoſitione exhibentur.

Cor. 1. Si Numeratorum M, N, O, P, &c. differentia vel prima, vel ſecunda, vel alia quædam detur, terminis omnibus poſt primos aliquot in Serie A, B, C, D, &c. evaneſcentibus, Differentiæ b, c, d, &c. tandem incurrent in Progreſſionem Geometricam in ratione 1 ad q. Exempli gratiâ, ſi detur Numeratorum M, N, O, P &c differentia prima B, erunt c, d, &c. in ratione continuâ Geometricâ 1 ad q; ut conſtat per ipſorum valores $qhA + hB$, $qqhA + qhB$, &c. exiſtentibus $C = 0 = D =$ &c.

Cor. 2. Ordo autem primæ differentiarum B, C, D, &c. quæ hoc modo evaneſcunt, idem eſt ac ordo differentiæ vel b, vel c, &c. unde incipit Progreſſio illa Geometrica. Sic ſi $B = 0 = C =$ &c. erunt b, c, d, &c. in Progreſſione Geometricâ; ſi $C = 0 = D =$ &c. erunt c, d, &c. in Progreſſione Geometricâ. Et ſic porrò.

Lemma 6.

Iiſdem poſitis ſit r terminus unde incipit Progreſſio Geometrica in Serie differentiarum b, c, d, &c. & per

$p+1$ deſignetur ordo Termini in Serie $\frac{\alpha}{h}, \frac{\beta}{h^2}, \frac{\gamma}{h^3}, \frac{\delta}{h^4}$, &c. Tum Terminus ille deſignabitur per fractionem cujus Denominatore exiſtente h^{p+1} Numerator eſt

$$a+bp+cp\times\frac{p-1}{2}+dp\times\frac{p-1}{2}\times\frac{p-2}{3}+\&c.+\frac{r}{q^n}$$

$$\times\overline{h^p-1-qp-q^2p\times\frac{p-1}{2}-q^3p\times\frac{p-1}{2}\times\frac{p-2}{3}-\&c.}$$

nempe per n deſignato ordine differentiæ evaneſcentis in Serie B, C, D, &c. ut & Numero terminorum $a+bp$, &c. item terminorum $-1-qp$, &c.

Demonſtratio. Per *Lemma* 1. Termini iſtius Numerator exhibetur per formulam

$$a+bp+cp.\frac{p-1}{2}+dp\times\frac{p-1}{2}.\frac{p-2}{3}+\&c.(p+1$$

ſubeunte vices x in Lemmate iſto)

Ergò ſi ſit, *ex. gr.* $n=2$, per *Lemm.* 5. *Cor.* 2. erunt c, d, &c. in ratione continuâ 1 ad q. Numerator itaque in hoc caſu eſt

$$a+bp+cp\times\frac{p-1}{2}+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+cq^2p$$

$\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}+$ &c. Sed ſi termini $cp\times\frac{p-1}{2}+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+$ &c. ducantur in $\frac{q^2}{c}$, & productui addantur termini $1+qp$, prodibit Series quà exprimitur binomii $1+q$ dignitas $\overline{1+q}|^p=h^p$. Ergo productum illud æquale eſt h^p-1-qp; adeoque termini $cp\times\frac{p-1}{2}+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+$ &c. $=\frac{c}{q^2}\times\overline{h^p-1-qp}$. Quo pacto Numerator fit $a+bp+\frac{c}{q^2}\times\overline{h^p-1-qp}$, exiſtentibus duobus terminis $a+bp$, ut & duobus $-1-qp$, juxta ſenſum Propoſitionis, quoniam $n=2$. Atque eadem eſt demonſtratio in aliis caſibus. De Denominatore verò per ſe ſatis conſtat.

Prop.

Prop. IX. *Prob.*

Invenire ſummam quotlibet terminorum Seriei cujuſvis $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, &c. cujus terminorum Denominatores conſtituunt progreſſionem quamlibet Geometricam h, h^2, h^3, h^4, &c. Numeratores autem ſunt quantitates differentiâ aliquâ conſtanti gaudentes.

Solutio Sunto Numeratorum M, N, O, P, &c. primus A, prima differentiarum primarum B, prima ſecundarum C, prima tertiarum D, & ſic porrò; & ſit ipſorum A, B, C, D, &c. numerus n, atque $h - 1 = q$, Tum fiat $a = A\ (= M)$ $b = hA + B$, $c = qhA + hB + C$, $d = q^2hA + qhB + hC + D$, &c. ut ſint tot termini $a, b, c\ d$, &c, quot ſunt unitates in $n + 1$. Terminorum iſtorum ultimus dicatur r, atque per $p + 1$ deſignetur numerus terminorum $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, &c. quorum ſumma requiritur; Dico ſummam illam exhiberi per fractionem, cujus Denominatore exiſtente h^{p+1}, Numerator eſt

$$\underline{a + bp + cp\times\frac{p-1}{2} + dp\times\frac{p-1}{2}\times\frac{p-2}{3} + \&c. + \frac{r}{q^n}}$$
$$\underline{\times h^p - 1} - \underline{qp - q^2p\times\frac{p-1}{2}} - q^3p\times\frac{p-1}{2}\times\frac{p-2}{3} - \&c.$$
$$- q^{n-1}p\times\frac{p-1}{2}\times \&c.$$

Demonſtratio. Nam (per *Lem.* 6.) per hanc formulam repræſentatur terminus ordine $p + 1$ Seriei $\frac{\alpha}{h}, \frac{\beta}{h^2}, \frac{\gamma}{h^3}, \frac{\delta}{h^4}$, &c. qui terminus (per conſtructionem *Lemmatis* 5.) æqualis eſt aggregato terminorum numero $p + 1$ Seriei propoſitæ $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$. *Q. E. D.*

Ex. 1. Invenienda ſit ſumma novem terminorum Seriei $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}$, &c. Sunt in hoc caſu $h = 2$, $q\ (= h - 1) = 1$, $p + 1 = 9$, $p = 8$, $A = 1$, $B = 1$, $C = 0, = D = \&c.$ adeoque $n = 2$, (quoniam ſunt duo $A, B,$) Hinc fit $a\ (= A) = 1$, $b\ (= hA + B = 2 \times 1 + 1) = 3$, $c\ (= qhA + hB + C = 2 \times 1 + 2 \times 1 + 0) = 4 = r$, Adeoque per formulam fit ſumma quæſita

$$\frac{1 + 3 \times 8 + \frac{4}{1^2} \times \overline{2^3 - 1 - 1 \times 8}}{2^9} = \frac{1013}{512}.$$

Ex. 2. Quæratur ſumma ſex terminorum Seriei $1 \times 3 + 3 \times 3^2 + 6 \times 3^3 + 10 \times 3^4 + 15 \times 3^5 + 21 \times 3^6 + \&c.$ In hoc caſu ſunt $h = \frac{1}{3}$, $q = \frac{-2}{3}$, $p + 1 = 6$, $p = 5$, $A = 1$, $B = 2$, $C = 1$, $D = 0 = E = \&c.$ adeoque $n = 3$, atque $a = 1$, $b = \frac{1}{3} + 2 = \frac{7}{3}$, $c = \frac{-2}{9} + \frac{2}{3} + 1 = \frac{13}{9}$, $d = \frac{4}{27} - \frac{4}{9} + \frac{1}{3} = \frac{1}{27} = r$. Unde ſumma quæſita fit $= 19956$. ſive

$$\frac{1 + \frac{7}{3} \times 5 + \frac{13}{9} \times 5 \times \frac{4}{2} + \frac{-1}{8} \times \overline{\frac{1}{3^5} - 1 + \frac{2}{3} \times 5 - \frac{4}{9} \times 5 \times \frac{4}{2}}}{\overline{\frac{1}{3}}\Big|^{6}}$$

Cor. 1. Ejuſdem Seriei, à termino primo $\frac{M}{h}$ in infinitum continuatæ, ſumma exhibetur per formulam ſimpliciſſimam $\frac{A}{\overline{h-1}} + \frac{B}{\overline{h-1}|^2} + \frac{C}{\overline{h-1}|^3} + \frac{D}{\overline{h-1}|^4}$ &c.

Cor. 2. Si $h = 2$, Seriei totius in infinitum continuatæ ſumma habetur ſolâ additione terminorum A, B, C, D, &c. Et hæc ſumma eadem eſt ac ſumma lineæ erectæ reſpondentis termino primo A, in Triangulo Arithmetico, cujus lineam tranſverſam occupant Numeratores *M, N, O, P, &c.* Quod ſacile conſtat ex contemplatione Trianguli. Si itaque ſuerint *M, N O, &c.* Numeri figurati cujuſvis ordinis *n*, ſumma Seriei $\frac{M}{2} + \frac{N}{4} + \frac{O}{8} + \frac{P}{16} + \&c.$ æqualis erit Numeri binarii dignitati $\overline{2}|^{n-1}$. Sic Series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \&c. = 2^{1-1} = 1$, ut vulgò notum; Series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \&c. = 2^{n-1} = 2$; Series $\frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \&c. = 2^{3-1} = 2^2 = 4$, & ſic porrò.

Scholium. Celeb. D. *Jac. Bernoulli*, in Tractatu ſuo de Seriebus infinitis, ſolvit illud Problema. "Invenire "ſummam Seriei infinitæ Fractionum quarum Denomi"natores creſcunt in Progreſſione quacunque Geome"tricâ, Numeratores verò progrediuntur vel juxta Nu"meros naturales, 1, 2, 3, 4, *&c.* vel Trigonales 1, "3, 6, 10, *&c.* vel Pyramidales 1, 4, 10, 20, *&c.* "aut juxta Quadratos 1, 4, 9, 16, *&c.* aut Cubos 1, "8, 27, 64, *&c.* eorumve multiplices." Ipſius ſolutionem conſulat Lector. Aliam verò, & quidem multo generaliorem invenit D. *Nic. Bernoulli* illius Nepos, eamque (poſtquam ei hæc miſeram, ſed ſine demonſtratione) mecum communicare dignatus eſt, in epiſtolâ datâ 18° *Septembris* 1715, miris quidem inventis refertiſſimâ, qualibus me crebro dignatur vir Doctiſſimus. De hoc vero Problemate ſic ſcribit. "Pour la ſomme "d'un nombre determiné *n* de termes de la ſuitte de "voſtre Theoreme 7. [*Corollarium* primum eſt hujus Propoſitionis] j'ay trouvé cette formule $\frac{1}{m^n} \times$

$$\overline{\times \frac{m-1}{m-1} a + \frac{A-n}{m-1} b + \frac{B-n.\frac{n-1}{2}}{m-1} c + \frac{C-n.\frac{n-1}{2}.\frac{n-2}{3}}{m-1} d}$$

" + *&c.* ou les Lettres *A, B, C, &c.* marquent

“ les Coefficients des termes immediatement prece-
“ dents. Et en mettant dans cette formule $p + 1$
“ pour n, h^m pour m, & en multipliant tout encore
“ par c^{m-1}, on a la ſolution de voſtre *Prob.*
“ IX^{me}”. Et me monuit Vir peritiſſimus hanc ſuam formulam generalem in noſtram particularem (*Cor.* 1. hujus propoſitionis) migrare quando $n = \infty$; quippe tum evaneſcunt $1, n, n.\frac{n-1}{2}, n.\frac{n-1}{2}.\frac{n-2}{3}$, *&c* reſpectu ipſorum m^n, A, B, C, *&c.* adeo ut Series in eo caſu ſit $\frac{1}{m-1}a + \frac{A}{m-1}b + \frac{B}{m-1}c +$ *&c.* quæ omnino coincidit cum noſtrâ $\frac{a}{m-1} + \frac{b}{\overline{m-1}|^{2}} + \frac{c}{\overline{m-1}|^{3}} +$ *&c.*

Adhuc aliam hujus Problematis ſolutionem, & quidem ab hiſce admodum diverſam, invenit D. *Taylor* ope Methodi ſuæ Incrementorum. Viri doctiſſimi rogatu, ad eum miſeram formulam meam ſecundam pro ſolutione Problematis II^{di}, item formulas alias ſpectantes ad Propoſitiones tertiam, quartam & quintam, ſed ſine demonſtrationibus: quippe non dubitabam quin Vir acutiſſimus, atque ipſe Methodi iſtius Incrementorum Inventor, hiſce, vel ſaltem paribus inveniendis par eſſet. Reſcripſit ſe harum ſolutiones inveniſſe, & ſimul alia quædam communicavit ad hujus methodi profectum multum facientia, quæ jam noſtro hortatu inductus hiſce ſubjungere dignatur.

Appendix

APPENDIX

Quâ methodo diverſâ eadem materia tractatur:

Auctore Brook Taylor, *LL. D. R. S. Secr.*

HOrtatu Viri Clariſſ. cui nos innumeris officiis devinctiſſimos eſſe libenter fatemur, ſequentes jam Propoſitiones exhibemus, quas quidem in aliam occaſionem reſervandas eſſe decreviſſemus, ni æquum viſum fuiſſet parendum eſſe imperio amici qui, dum Propoſitiones quaſdam præcedentes ſuas olim nobis inveſtigandas propoſuit, earum inveniendarum occaſionem dedit.

Definitiones.

1. Quantitatis cujuſvis variabilis valorem præſentem deſigno literâ ſimpliciter ſcriptâ, ut x; valores præcedentes diſtinguo lineolis eidem literæ ex parte ſuperiori poſitis, ſequentes lineolis ex parte inferiori ſcriptis. Ut vi hujus Definitionis ſint $\overset{\prime\prime}{x}$, $\overset{\prime}{x}$, x, $\underset{\prime}{x}$, $\underset{\prime\prime}{x}$, ejuſdem variabilis valores quinque continui, exiſtente x valore præſenti, $\overset{\prime}{x}$ proximè præterito, $\overset{\prime\prime}{x}$ ſecundò præterito; $\underset{\prime}{x}$ proximè, atque $\underset{\prime\prime}{x}$ ſecundò futuro. Et ſic de aliis. Ad eundem modum ſunt interpretandæ lineolæ quæ incrementis apponuntur. Sic ſunt $\overset{\prime\prime}{\underset{\cdot\cdot}{x}}$, $\overset{\prime}{\underset{\cdot\cdot}{x}}$, $\underset{\cdot\cdot}{x}$, $\underset{\prime\cdot\cdot}{x}$, $\underset{\prime\prime\cdot\cdot}{x}$, ipſius $\underset{\cdot\cdot}{x}$ valores quinque continui; ut ſit $\overset{\prime\prime}{\underset{\cdot\cdot}{x}}$ incrementum

ſecundum ipſius $\overset{\prime\prime}{x}$, ſit $\underset{\cdot\cdot}{\overset{\prime}{x}}$ incrementum ſecundum ipſius $\overset{\prime}{x}$. Et ſic de aliis.

Cor. Vi hujus Definitionis, $\overset{\prime}{x} + \underset{\cdot}{\overset{\prime}{x}} = x$, $x + \underset{\cdot}{x} = \underset{,}{x}$, $\underset{\cdot}{x} + \underset{\cdot\cdot}{x} = \underset{,\cdot}{x}$. Et ſic de aliis hujuſmodi.

Quando uſu venit ut variabilis quantitas, puta x, ſpectanda ſit tanquam Incrementum, ejus Integrale deſigno literâ inter uncos [ ] incluſâ. Iſtius etiam Integralis $[x]$ Integrale (vel ipſius x Integrale ſecundum,) deſigno numero binario uncorum priori ſuperimpoſito, ut $\overset{2}{[x]}$. Iſtius etiam Integralis Integrale (vel ipſius x Integrale tertium,) ad eundem modum deſigno numero ternario, ut $\overset{3}{[x]}$. Et ſic deinceps. Unde vi hujus Definitionis conſtituunt $\overset{3}{[x]}$, $\overset{2}{[x]}$, $[x]$, x Seriem terminorum, quorum quilibet eſt ipſum immediatè præcedentis incrementum primum, ut ſit $\overset{2}{[x]} = \overset{3}{[\underset{\cdot}{x}]}$, $[x] = \overset{2}{[\underset{\cdot}{x}]}$, $x = [\underset{\cdot}{x}]$.

Lemma.

Facti xv ex Multiplicatione duorum variabilium x & v, incrementum eſt $\underset{\cdot}{x}v + \underset{,}{x}\underset{\cdot}{v}$.

Nam auctis variabilibus per propria incrementa, ſit novum productum $\overline{x + \underset{\cdot}{x}} \times \overline{v + \underset{\cdot}{v}}$, ſive $xv + \underset{\cdot}{x}v + \overline{x + \underset{\cdot}{x}} \times \underset{\cdot}{v}$, hoc eſt $xv + \underset{\cdot}{x}v + \underset{,}{x}\underset{\cdot}{v}$ (pro $x + \underset{\cdot}{x}$ ſcripto $\underset{,}{x}$ *per Def.* 1.)

Unde dempto priori producto xv, reſtat Incrementum $\underset{\cdot}{x}v + \underset{,}{x}\underset{\cdot}{v}$.

Prop. I. *Theor.*

Ejuſdem Facti xv Incrementum, vel primum, vel ſecundum, vel tertium, vel aliud quodvis, cujus ordo deſignatur per ſymbolum n, exhibetur per formulam hanc generalem

$$\underset{n}{x}v + n\,\underset{'n-1}{x}\,\underset{\cdot}{v} + n\times\frac{n-1}{2}\,\underset{''n-2}{x}\,\underset{\cdot}{v} + n\times\frac{n-1}{2}\times\frac{n-2}{3}\,\underset{'''n-3}{x}\,\underset{\therefore}{v} + \text{\&c.}$$

In hâc formulâ hæc ſunt obſervanda, 1^{mo} Terminorum numeri coefficientes 1, n, $n\times\frac{n-1}{2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}$ *&c.* iidem ſunt ac in binomii dignitate n. 2^{do} Numeri n, $n-1$, $n-2$, $n-3$, *&c.* ipſis x infraſcripti deſignant numeros punctorum quibus definiuntur Incrementa. 3^{tio} Lineolæ $,$ $,,$ $,,,$ *&c.* ipſis x infraſcriptæ, interpretandæ ſunt per Def. 1. 4^{to}. In quovis Termino numerus punctorum ipſis x & v ſimul infraſcriptorum, eſt n. Sit *v.g.* $n = 4$: tum per formulam, ipſius xv incrementum quartum prodit $\underset{::}{x}v + 4\,\underset{,\therefore}{x}\,\underset{\cdot}{v} + 6\,\underset{,,\cdot\cdot}{x}\,\underset{\cdot\cdot}{v} + 4\,\underset{,,,\cdot}{x}\,\underset{\therefore}{v} + \underset{,,,,}{x}\,\underset{::}{v}$.

Theorema hoc generale demonſtrari poteſt per Inductionem, incrementis continuò ſumptis juxta formam in Lemmate præcedenti traditam. Sed & collectâ formâ Seriei ex hujuſmodi calculo, Theorema etiam demonſtrari poteſt per Methodum Incrementorum, ad eum modum cujus ſpecimen mox dabimus in demonſtratione Propoſitionis tertiæ.

Prop. II. *Theor.*

Ipſius xv Integrale primum $[xv]$ exhibetur per Seriem $[x]v - \overset{2}{[x]}_{,}\,\underset{\cdot}{v} + \overset{3}{[x]}_{,,}\,\underset{\cdot\cdot}{v} - \overset{4}{[x]}_{,,,}\,\underset{\therefore}{v}$ *&c.*

Series autem ita terminatur, ut ſit $[xv] = [x]\,v$

$$-\left[[\underset{\prime}{x}]\underset{.}{v}\right]=[x]\,v-\overset{2}{[\underset{\prime}{x}]}\underset{.}{v}+\left[\overset{2}{[\underset{\prime\prime}{x}]}\underset{..}{v}\right]=\&c.$$

Nam ſumendo incrementa reſtituitur propoſitum $x v$.

Cor. 1. Datis duobus ex iſtis $[x]$, $[x v]$, $\left[[\underset{\prime}{x}]\underset{.}{v}\right]$, datur tertium. Item datis tribus ex iſtis $[x]$, $\overset{2}{[\underset{\prime}{x}]}$, $[x v]$, $\left[\overset{2}{[\underset{\prime\prime}{x}]}\underset{.}{v}\right]$, datur quartum, Et ſic porrò.

Cor. 2. Si $\underset{.}{v}=0$, datur $[x v]$ ex dato $[x]$. Si $\underset{..}{v}=0$ datur $[x v]$ ex datis duobus $[x]$, & $\overset{2}{[x]}$, Si $\underset{\cdot\cdot\cdot}{v}=0$, datur $[x v]$, ex datis tribus $[x]$, $\overset{2}{[x]}$, $\overset{3}{[x]}$. Et ſic porrò.

Ex. 1. Sit exemplum hujus formulæ in inventione Integralis ipſius $\frac{v}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}$, dato nempe $\underset{.}{z}$, atque exiſtente $\underset{\cdot\cdot\cdot}{v}=0$, qui caſus eſt ſpecialis Propoſitionis ſecundæ Tractatus præcedentis D$^{\text{ni}}$ *Monmort*. Facto itaque $x=\frac{1}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}$, ſunt $[x]=\frac{-1}{3\underset{.}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}}$, $\overset{2}{[\underset{\prime}{x}]}=\frac{1}{2\underset{.}{z}\times 3\underset{.}{z}\underset{\prime}{z}\underset{\prime\prime}{z}}$, atque $\overset{3}{[\underset{\prime\prime}{x}]}=\frac{-1}{1\underset{.}{z}\times 2\underset{.}{z}\times 3\underset{.}{z}\underset{\prime\prime}{z}}$. Unde per formulam fit $[x v]$, hoc eſt

$$\left[\frac{v}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}\right]=-\frac{v}{3\underset{.}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}}=-\frac{\underset{.}{v}}{2\underset{.}{z}\times 3\underset{.}{z}\underset{\prime}{z}\underset{\prime\prime}{z}}-\frac{\underset{..}{v}}{1\underset{.}{z}\times 2\underset{.}{z}\times 3\underset{.}{z}\underset{\prime\prime}{z}}.$$

Ex. 2. Sit aliud exemplum in inventione Integralis ipſius na^{z}, ubi eſt $\underset{\cdot}{z} = 1$, atque datur a, Tum pro x ſumpto a^{z}, & pro v ſumpto n, fit $\underline{\underset{,}{x} = a^{\underset{,}{z}}}$ hoc eſt $\underset{,}{x} = ax$, ſeu $x + \underset{\cdot}{x} = ax$, adeoque $\underset{\cdot}{x} = \overline{a-1}\,x$, atque $x = \frac{\underset{\cdot}{x}}{a-1}$. Regrediendo itaque ad Integralia fit $[x] = \frac{x}{a-1}$; item $\overset{2}{[x]} = \frac{[x]}{a-1} = \frac{x}{\overline{a-1}|^{2}}$, item $\overset{3}{[x]} = \frac{x}{\overline{a-1}|^{3}}$; & ſic porrò. Adeoque (quoniam $\underset{,}{x} = ax$,) ſunt $[x] = \frac{x}{a-1}$, $[\underset{,}{x}] = \frac{ax}{\overline{a-1}|^{2}}$, $[\underset{,,}{x}] = \frac{a^{2}x}{\overline{a-1}|^{3}}$, &c. Unde per formulam prodit $[na^{z}] = \frac{a^{z}n}{a-1} - \frac{a^{z+1}\underset{\cdot}{n}}{\overline{a-1}|^{2}} + \frac{a^{z+2}\underset{\cdot\cdot}{n}}{\overline{a-1}|^{3}}$ &c.

In hoc exemplo continetur Solutio Problematis, de quo agit D^nus^ de *Monmort* in Propoſitione nona. Coincidit autem formula cum ea quam exhibet ille in Corollario primo ejuſdem Propoſitionis.

Scholium. Poſſunt etiam ex hâc formulâ alii derivari valores Integralis quæſiti, pro vario modo quo interpretantur Incrementi propoſiti factores. Sic in exemplo ſecundo integrale ipſius na^{z} exhiberi poteſt per formulam $a^{z}[n] - \overline{a-1}\,a^{z}\overset{2}{[\underset{,}{n}]} + \overline{a-1}|^{2}\,a^{z}\overset{3}{[\underset{,,}{n}]}$ — &c. pro x nempe ſumpto n, & pro v ſumpto a^{z}. Sed de his fortaſſe aliâ occaſione fuſius dicemus.

Prop. III. *Theor.*

Ejuſdem xv Integrale, vel primum, vel ſecundum, vel tertium, vel aliud quodvis cujus ordo deſignatur ſymbolo n, exhibetur per Seriem in hâc formâ generali prodeuntem $\overset{n}{[xv]} = \overset{n}{[x]}\,v - n\,\overset{n+1}{[\underset{,}{x}]}\,\underset{\cdot}{v} +$

$$+\, n \times \frac{n+1}{2}\, \overset{n+2}{[\underset{\prime\prime}{x}]}\, \underset{\cdot\cdot}{v} - n \times \frac{n+1}{2} \times \frac{n+2}{3}\, \overset{n+3}{[\underset{\prime\prime\prime}{x}]}\, \underset{\because}{v} + \text{\&c.}$$

Collectâ formâ Seriei ex Propoſitione præcedenti, Coefficientes $1, -n, n \times \frac{n+1}{2}, -n \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. ſic inveniuntur per Methodum Incrementorum. Pone

$$\overset{n}{[xv]} = A\overset{n}{[x]}v + B\overset{n+1}{[\underset{\prime}{x}]}\underset{\cdot}{v} + C\overset{n+2}{[\underset{\prime\prime}{x}]}\underset{\cdot\cdot}{v} + D\overset{n+3}{[\underset{\prime\prime\prime}{x}]}\underset{\because}{v} + \text{\&c.}$$

Tum aucto n incremento ſuo $\underset{\cdot}{n} = 1$, atque ipſis A, B, C, D, &c. incrementis ſuis contemporaneis $\underset{\cdot}{A}, \underset{\cdot}{B}, \underset{\cdot}{C}, \underset{\cdot}{D}$, &c. ut jam evadant $\underset{\prime}{n}, \underset{\prime}{A}, \underset{\prime}{B}, \underset{\prime}{C}, \underset{\prime}{D}$, &c. fiet novum Integrale (quod Integrale eſt ipſius $\overset{n}{[xv]}$,) $\overset{n+1}{[xv]} =$

$$\underset{\prime}{A}\overset{n+1}{[x]}v + \underset{\prime}{B}\overset{n+2}{[\underset{\prime}{x}]}\underset{\cdot}{v} + \underset{\prime}{C}\overset{n+3}{[\underset{\prime\prime}{x}]}\underset{\cdot\cdot}{v} + \underset{\prime}{D}\overset{n+4}{[\underset{\prime\prime\prime}{x}]}\underset{\because}{v} + \text{\&c.}$$

Hujus itaque Incrementum primum coincidere debet cum Integrali prius poſito. Sumptis ergo incrementis, fit

$$\overset{n}{[xv]} = \underset{\prime}{A}\overset{n}{[x]}v \begin{matrix} +\underset{\prime}{A} \\ +\underset{\prime}{B} \end{matrix} \overset{n+1}{[\underset{\prime}{x}]}\underset{\cdot}{v} \begin{matrix} +\underset{\prime}{B} \\ +\underset{\prime}{C} \end{matrix} \overset{n+2}{[\underset{\prime\prime}{x}]}\underset{\cdot\cdot}{v} \begin{matrix} +\underset{\prime}{C} \\ +\underset{\prime}{D} \end{matrix} \overset{n+3}{[\underset{\prime\prime\prime}{x}]}\underset{\because}{v} +$$

idem ac Integrale prius poſitum. Itaque terminos homologos inter ſe comparando fit 1^{mo} $\underset{\prime}{A} = A$. Unde eſt A datum quid. Sed ubi $n = 0$, eſt $A = 1$, ergo $A = 1$. 2^{do}. $B = \underset{\prime}{B} + \underset{\prime}{A}$, hoc eſt $B = B + \underset{\cdot}{B} + 1$, ſeu $B = -1 = -n$. Ergo regrediendo ad Integralia, fit $B = -n + a$. Sed ubi $n = 0$, eſt $B = 0$. Ergo $a = 0$, atque $B = -n$. 3^{tio}. $C = \underset{\prime}{C} + \underset{\prime}{B}$, hoc eſt $\underset{\cdot}{C} = \underset{\prime}{n}$. Regrediendo itaque ad Integralia fit $C = \frac{n\underset{\prime}{n}}{2} + b$. Sed ubi $n = 0$, eſt $C = 0$. Ergo $b = 0$, atque $C = \frac{n\underset{\prime}{n}}{2}$, hoc eſt, $n \times \frac{n+1}{2}$. 4^{to}. Ad eundem modum invenitur $D = -n \times \frac{n+1}{2} \times \frac{n+2}{3}$. Et ſic pergendo inveniuntur cæteri Coefficientes.

Scholium. 1. In hâc Propoſitione comparatâ cum Propoſitione primâ, cernitur ſingularis quædam relatio Incrementa inter & Integralia. Ut enim in Arithmeticâ vulgari, Multiplicatio & Diviſio ſunt invicem ita contrariæ, ut ſi Multiplicatio deſignetur per Indicem affirmativum, Diviſio deſignabitur per Indicem cum ſigno negativo; ſic etiam in Methodo Incrementorum, ſi Incrementum deſignetur per Indicem affirmativum, Index negativus Integrale ſiſtet. Sic in Propoſitione primâ, ſi pro n ſumatur Numerus binarius 2, per formulam exhibebitur ipſius xv incrementum ſecundum, nempe $\underset{\cdot\cdot}{x}v + 2\underset{\prime\cdot}{x}\underset{\cdot}{v} + \underset{\prime\prime}{x}\underset{\cdot\cdot}{v}$; Sed ſi pro n ſumatur numerus negativus -2, ut jam quæratur ipſius xv incrementum (ita loqui liceat) negativè ſecundum, (quod idem eſt ac Integrale ſecundum) prodeunt coefficientes iidem ac ſi ſumatur n affirmativè in Propoſitione præſenti: atque interpretatis inſuper ipſis $\overset{-2}{x}$, $\overset{\prime-3}{x}$, $\overset{\prime\prime-4}{x}$, &c. per $\overset{2}{[x]}$, $\overset{3}{[\underset{\prime}{x}]}$, $\overset{4}{[\underset{\prime\prime}{x}]}$, &c. Series fit omnino eadem ac per Propoſitionem præſentem prodit, ubi quæritur Integrale ſecundum.

2. Ex his autem formulis quaſi ſuâ ſponte procedunt formulæ Propoſitionum undecimæ atque duodecimæ Libri de Methodo Incrementorum. Nam pro

incrementis ſcribe Fluxiones, atque evaneſcentibus incrementis fiant jam omnes $x, x_{\prime}, x_{\prime\prime}, x_{\prime\prime\prime}$, &c. inter ſe æquales, atque migrabit ſtatim hæc Propoſitio ſecunda in illam undecimam, atque præſens tertia in illam duodecimam. Quod quidem exemplum ſatis inſigne eſt Methodi *Newtonianæ*, quâ colligit ille rationes Fluxionum ex rationibus ultimis Incrementorum evaneſcentium, vel ex primis naſcentium.

Additamentum.

PRæcedentium impreſſioni intentus dum Typothetarum erroribus corrigendis do operam, atque eâ occaſione in animo illa ſæpius revolvo, ſubiit Artificium illud quo jam olim uſus eſt D. *Jac Bernoulli* in inventione quarundam Serierum, ope Progreſſionis Harmonicæ cujus meminit D. de *Monmort* in *Scholio* 6. *Prop.* V. præcedente commodè etiam applicari poſſe ad inventionem ipſius *Monmortii* Propoſitionum $2^{dæ}$, $3^{tiæ}$, $4^{tæ}$, $5^{tæ}$, atque id genus aliarum aliquanto fortaſſe generaliorum. Hoc in ſequentibus paucis oſtendiſſe, credebam Lectori non fore ingratum.

Theorema.

Sit Progreſſio Arithmetica p, $p+n$, $p+2n$, &c. cujus termini ſinguli ſucceſſivè deſignentur per x, & ſunto b, c, d, &c. quivis multiplices differentiæ datæ n terminorum Progreſſionis iſtius Arithmeticæ. Sint A, B, C, D, &c. Numeri quilibet dati, & conſtituantur fractiones quotvis $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, $\frac{D}{x+d}$, &c. Pro x ſucceſſive ſcriptis valoribus ſuis p, $p+n$, $p+2n$, &c.

[illegible]

ex harum fractionum quâlibet, oritur Series Harmonicè proportionalium Sic *v g.* ex fractione primâ $\frac{A}{p}$, oritur Series $\frac{A}{p}$, $\frac{A}{p+n}$, $\frac{A}{p+2n}$, *&c.* Dico quod aggregatum quotlibet hujuſmodi Serierum in infinitum continuatarum in terminis numero finitis exhiberi poteſt, ſi modo fuerit numeratorum *A*, *B*, *C*, *D*, *&c.* aggregatum æquale nihilo. Duobus exemplis hoc fiet manifeſtum.

Ex. Sint duæ tantùm fractiones $\frac{A}{x}$, atque $\frac{-A}{x+3n}$, exiſtente $b = 3n$. Scribantur Series harmonicæ ex his formulis ortæ, eo ordine, ut termini, in quibus ſunt denominatores æquales, ſibi invicem reſpondeant, & collectis ſummis terminorum homologorum, prodibit aggregatum Serierum in terminis numero finitis, ut in calculo appoſito videre eſt.

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+\frac{A}{p+3n}+\frac{A}{p+4n}+\&c. = \text{Seriei ortæ ex } \frac{A}{x}$$

$$+\frac{-A}{p+3n}+\frac{-A}{p+4n}+\&c. = \text{Seriei ex } \frac{-A}{x+3n}$$

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+0+0+\&c. = \text{Aggreg. Serierū.}$$

Ex. 2. Sint tres fractiones $\frac{A}{x}$, $\frac{B}{x+2n}$, $\frac{C}{x+3a}$, exiſtentibus $b = 2n$, $c = 3n$, atque $A + B + C = 0$. In hoc caſu Calculus ſic ſe habet.

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+\frac{A}{p+3n}+\ldots+\&c. = \text{Seriei ortæ ex } \frac{A}{x}$$

$$+\frac{B}{p+2n}+\frac{B}{p+3n}+\ldots\ldots+\&c. = \text{Seriei ex } \frac{B}{x+2n}$$

$$+\frac{C}{p+3n}+\ldots\ldots+\&c. = \text{Seriei ex } \frac{C}{x+3n}$$

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A+B}{p+2n}+\frac{A+B+C=0}{p+3n}+\&c. = \text{Aggregato Serierum.}$$

Ubi etiam prodit aggregatum Serierum in terminis numero finitis, nempe $\frac{A}{p}+\frac{A}{p+n}+\frac{A+B}{p+2n}$, ob Numeratorum A, B, C, aggregatum æquale nihilo. Et ad eundem modum demonſtratur Theorema in aliis caſibus quibuſvis.

Cor. 1. Ex his principiis derivari poſſunt innumeræ Series in infinitum continuatæ, in terminis tamen numero finitis ſummabiles.

Caſ. 1. Sint $\frac{A}{x}$ & $\frac{-A}{x+b}$ formulæ duarum Serierum harmonicarum quarum aggregatum prodit in terminis numero finitis per ſuperius demonſtrata, Tum, formulis iſtis in unam ſummam collectis, fit $\frac{Ab}{x\times\overline{x+b}}$ formula Seriei ſummabilis. Sint *v. gr.* $A=\frac{1}{6}$, $p=1$, $n=2$, atque $b=3n=6$. Tum formulæ Serierum harmonicarum erunt $\frac{1}{6x}$, & $\frac{-1}{6\times\overline{x+6}}$, formula Seriei compoſitæ ſummabilis erit $\frac{1}{x\times\overline{x+6}}$, Serie illa exiſtente $\frac{1}{1\times7}+\frac{1}{3\times9}+\frac{1}{5\times11}+\frac{1}{7\times13}+$ *&c.* atque ſumma Seriei, per calculum in præmiſſis demonſtratum, erit $\frac{1}{6\times1}+\frac{1}{6\times3}+\frac{1}{6\times5}$. Sint tres formulæ Serierum harmonicarum $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, (exiſtente $A+B+C=0$, ut ſit Serierum aggregatum finitum per præmiſſa.) Tum formulis in unam ſummam collectis fit

$$\frac{A\times\overline{x+b}\times\overline{x+c}+B\times x\times\overline{x+c}+C\times x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}},$$

ſeu (terminis revocatis ad formam factorum x, $x\times\overline{x+b}$, $x\times\overline{x+b}\times\overline{x+c}$,)

$$\frac{Acb+\overline{Ac+\overline{c-b}\,B}\times x+\overline{A+B+C}\times x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}},$$

hoc eſt

(ob $A + B + C = 0$) $\frac{Acb + Ac + B \times \overline{c - b} \times x}{x \times \overline{x + b} \times \overline{x + c}}$, formula Seriei ſummabilis. Si quatuor ſint Fractiones $\frac{A}{x}$, $\frac{B}{x + b}$, $\frac{C}{x + c}$, $\frac{D}{x + d}$, (exiſtente $A + B + C + D = 0$) ad eundem modum invenietur formula Seriei ſummabilis

$$\frac{Abcd + \overline{Acd + B \times \overline{c - b} \times \overline{d - b}} \times x + \overline{Ad + B \times \overline{d - b} + C \times \overline{d - c}} \times x \times \overline{x + b}}{x \times \overline{x + b} \times \overline{x + c} \times \overline{x + d}}$$

Et ſic pergere licet ad formulas adhuc magis compoſitas.

Caſ. 2. Et ſi plures ſint formulæ Serierum hujuſmodi ſummabilium, quarum denominatorum factores excerpantur ex diverſis progreſſionibus Arithmeticis, ex iſtarum formularum quotvis in unam ſummam additione, conficietur formula nova Seriei ſummabilis. Sint *e. gr.* formulæ duæ Serierum ſummabilium $\frac{1}{x \times \overline{x + 3}}$ & $\frac{1}{z \times \overline{z + 2}}$, excerptis x ex Progreſſione Arithmeticâ 1, 2, 3, 4, *&c.* z ex Progreſſione Arithmeticâ 1, 3, 5, *&c.* Tum ex his formulis in unam ſummam collectis fiet formula nova $\frac{z \times \overline{z + 2} + x \times \overline{x + 3}}{x \times \overline{x + 3} \times z \times \overline{z + 2}}$, vel, (expoſito z per x & numeros datos) $\frac{\overline{2x - 1} \times \overline{2x + 1} + x \times \overline{x + 3}}{x \times \overline{x + 3} \times \overline{2x - 1} \times \overline{2x + 1}}$

Cor. 2. Hinc omnis Series in infinitum continuata ſummabilis eſt, cujus termini deſignantur per Fractionem, cujus denominatoris factores excerpuntur ex datâ quâlibet Progreſſione Arithmeticâ, numerator autem eſt multinomium, cujus dimenſiones ſunt ad minimum binario pauciores, quam ſunt dimenſiones Denominatoris Nam omnis hujuſmodi fractio reſolvi poteſt in tot fractiones ſimplices, quot ſunt dimenſiones (hoc eſt, quot ſunt factores) Denominatoris, quarum numeratorum aggregatum eſt nihil. Sit exempli gratiâ,

formula oblata $\frac{\alpha + \beta x + \gamma x \times \overline{x+b}}{x \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}}$. Pone hanc formulam æquari aggregato fractionum $\frac{A}{x} + \frac{B}{x+b} + \frac{C}{x+c} + \frac{D}{x+d}$. Tum fractionibus iſtis in unam ſummam collectis fiet $Abcd + \overline{Acd + B\overline{c-b} \times \overline{d-b}} \times x$
$+ \overline{Ad + B \times \overline{d-b} + C \times \overline{d-c}} \times x \times \overline{x+b}$
$+ \overline{A+B+C+D} \times x \times \overline{x+b} \times \overline{x+c}$ applicatum ad $x \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} = \frac{\alpha + \beta \;\; + \gamma x \times \overline{x+b}}{x \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}}$,

Unde per comparationem terminorum homologorum fit $Abcd = \alpha$, $Acd + B \times \overline{c-b} \times \overline{d-b} = \beta$, $Ad + B \times \overline{d-b} + C \times \overline{d-c} = \gamma$ $A + B + C + D = 0$. adeoque $A = \frac{\alpha}{bcd}$, $B = \frac{\beta - Acd}{\overline{c-b} \times \overline{d-b}}$,

$C = \frac{\gamma - Ad - B \times \overline{d-b}}{d-c}$, $D = -A - B - C$, Quo pacto formula oblata reſolvitur in fractiones ſimplices $\frac{\alpha}{bcdx}$ $+ \frac{\beta - Acd}{\overline{c-b} \times \overline{d-b} \times \overline{x+b}} + \frac{\gamma - Ad - B \times \overline{d-b}}{\overline{d-c} \times \overline{x+c}}$ $+ \frac{-A - B - C}{x+d}$, ex quibus ortarum Serierum aggregatum, hoc eſt, ſumma Seriei ortæ ex formulâ oblatâ $\frac{\alpha + \beta x + \gamma x \times \overline{x+b}}{x \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}}$, per jam dicta prodit in terminis numero finitis. Quod verò dimenſiones numeratoris in formulâ oblatâ, debeant eſſe binario ad minimum pauciores, quam ſunt dimenſiones Denominatoris, hinc conſtat quod in reductione fractionum $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, $\frac{D}{x+d}$, quilibet numerator A, B, C, D, ducitur in omnes denominatores excepto uno, nempe ſuo; unde prodeunt Numeratoris Dimenſiones unitate pauciores quam ſunt dimenſiones Denominatoris. Sed per æquationem $A + B + C + D = 0$ perit altiſſima dimenſio in numeratore; Unde ſuperſunt Numeratoris Dimenſiones ad minimum binario-pauciores quam ſunt dimenſiones Denominatoris. Ad hoc verò Corollarium revocari poſſunt D. de *Monmort* Propoſitiones 2^{da} & 5^{ta}.

Cor. 3. Item oblatâ formulâ juxta *Caſ.* 2. *Cor.* 1. adhuc magis compoſitâ, ex iidem principiis perſpici poteſt an ſit Series ſummabilis. Sint progreſſiones duæ Arithmeticæ 1, 3, 5, *&c.* 2, 4, 6, *&c.* quarum termini homologi deſignentur per x & z, & ſit formula Seriei oblata $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+2} \times z \times \overline{z+2}}$, vel (pro z ſcripto $x + 1$, & factoribus Denominatoris in ordinem coactis) $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$. Pone formulam hanc æquari aggregato formularum $\frac{P}{x \times \overline{x+2}}$, $\frac{Q}{\overline{x+1} \times \overline{x+3}}$, Serierum per ſuperius dicta ſummabilium, ut (formulis his noviſſimis in unam ſummam collectis) ſit $\frac{P \times \overline{x+1} \times \overline{x+3} + Q \times x \times \overline{x+2}}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$ ſeu

$$\frac{3P + \overline{4P + 2Q}x + \overline{P + Q}x^2}{x \times \overline{x+\ } \times \overline{x+2} \times \overline{x+3}} = \frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}.$$

Hinc comparando terminos homologos oriuntur æquationes $3P = \alpha$, $4P + 2Q = \beta$, $P + Q = \gamma$. Unde eliminatis P & Q per debitas operationes Analyticas, prodit æquatio $2\alpha - 3\beta + \gamma = 0$, qua definitur relatio quæ inter coefficientes α, β, γ intercedere debet, ut Series orta ex formulâ oblatâ $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$

ſit ſummabilis. Ad eundem modum ſi formulæ oblatæ Denominatoris factores excerpantur ex tribus Progreſſionibus Arithmeticis, invenientur duæ æquationes quibus definiantur relationes coefficientium Numeratoris, ut ſit Series ſummabilis. Si quatuor ſint Progreſſiones Arithmeticæ, Coefficientium relatio definietur per tres æquationes. Et ſic porrò. Et in hujuſmodi formulis ut ſint Series ſummabiles, hæc inſuper obſervanda ſunt, Primò ut Numeratorum dimenſiones ſint ad minimum binario pauciores quam ſunt dimenſiones Denominatorum, Deinde ut ex ſingulis Progreſſionibus Arithmeticis excerpantur ad minimum duo factores Denominatoris. Denique, quod ſi ſint duo vel plures factores Denominatoris inter ſe æquales, ponendum ſit tot etiam Progreſſiones Arithmeticas, ex quibus excerpuntur, eſſe inter ſe æquales. Præmiſſis attentius perpenſis, hæc obvia erunt. Ad hoc vero Corollarium facile revocantur D. de *Monmort* Propoſitiones 3^{tia} & 4^{ta}.

FINIS.

ERRATUM in N°. 352.

PAge 586, after the end of line 15, add *black Cloud, from behind which there iſſued a.*

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